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Integrated Fault Estimation and Fault Tolerant Control for Stochastic Systems with Brownian Motions

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SUMMARY

This paper presents an integrated robust fault estimation and fault tolerant control technique for stochastic systems subjected to Brownian parameter perturbations. Augmented system approach, unknown input observer method, and optimization technique are integrated to achieve robust simultaneous estimates of the system states and the means of faults concerned. Meanwhile, a robust fault tolerant control strategy is developed by using actuator and sensor signal compensation techniques. Stochastic linear time-invariant systems, stochastic systems with Lipschitz nonlinear constraint, and stochastic systems with quadratic inner-bounded nonlinear constraint, are respectively investigated, and the corresponding fault-tolerant control algorithms are addressed. Finally, the effectiveness of the proposed fault tolerant control techniques is demonstrated via the drive train system of a 4.8MW benchmark wind turbine, a three-tank system and a numerical nonlinear model.

KEY WORDS: Brownian motions; fault estimation; fault tolerant control; stochastically input-to-state stability

1. INTRODUCTION

Real-time industrial systems are unavoidably subjected to various malfunctions or unanticipated abnormal behaviors, which may result in unexpected repairing/maintenance cost and safety hazard. Therefore, there is a high demand to develop advanced fault diagnosis and

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fault tolerant control strategies to enhance reliability, safety, and availability in industrial processes. During the last decades, fruitful results on fault diagnosis and tolerant control approaches and their applications were reported, e.g., see [1-5].

Stochastic dynamics widely exist in many industrial processes such as nuclear systems, chemical processes, biological systems, thermal systems, wind energy conversion systems, and so forth. Therefore, researches on fault diagnosis and tolerant control for industrial processes with stochastic natures are well motivated, and some interesting results were reported for systems with various stochastic descriptions, such as white noises [6, 7], Markovian jump distributions [8, 9], non-Gaussian disturbances [10] and Brownian parameter perturbations [11, 12]. It is worthy to point out that stochastic systems formulated by Itô-type stochastic differential equations have attracted much attention recently [13, 14], owing to their flexibility to describe a wide range of stochastic processes.

Signal compensation is one of the powerful fault tolerant control strategies, which can eliminate the influences from the occurred faults to the system dynamics [15]. Before implementing signal compensation, the size and shape of the faults should be available, which can be obtained by using a variety of fault estimation/reconstruction techniques, such as augmented system approaches [15], adaptive observer techniques [16, 17], sliding mode observer methods [18, 19] and so forth. Among the fault estimation approaches above, the augmented system approach can provide a simultaneous estimate of the system states and the faults concerned, which can thus be utilized for signal compensation based tolerant control.

Uncertainties always exist in the system modelling processes; therefore, robustness against unknown uncertainties plays a key role in designing an effective fault estimation and tolerant control algorithm. Recently, robust fault estimation techniques have been developed by implementing fault estimation with additional observer techniques [20, 21] to reduce the influences from the uncertainties. Unknown input observer (UIO) technique is a robust design method for state estimation and fault detection by decoupling the influences from the unknown process uncertainties to the estimation error and the residual [22]. However, in many industrial models, process disturbances cannot be completely decoupled, which makes the conventional UIO methods invalid. Motivated by this, UIO techniques associated with optimization approaches have been developed in [23, 24] for systems corrupted by partially decoupled process disturbances, where the disturbances that cannot be decoupled were attenuated by using optimization techniques. In [25], by integrating augmented system approach and UIO techniques, a simultaneous state/fault estimation technique was developed for stochastic systems with quadratic inner-bounded nonlinear constraint, Brownian perturbations, and

partially decoupled process disturbances. It is noticed that the input-to-state stability captures the idea that bounded unknown inputs result in bounded system trajectories [26], which is useful for the stability analysis of the systems with unknown input uncertainties. To the best of our knowledge, no effort has been paid on robust fault tolerant control for stochastic Brownian systems with the aid of stochastic input-to-state stability theory.

Comparing to the existing works, the remarkable distinctions and contributions of this paper are summarized as follows:

- i) The concerned unknown input disturbances are partially decoupled rather than completely decoupled, which can meet practical requirement better.
- ii) The stochastic systems under investigation are represented by Itô-type stochastic differential equations, which can describe real dynamic processes more precisely but bring more challenges due to the Brownian motions.
- iii) UIO jointly with augmented system approach is employed to achieve state/fault estimation, and decouple a part of unknown inputs. Linear matrix inequality (LMI) optimization algorithm is then utilized to attenuate the remaining part of the unknown inputs that cannot be decoupled by UIO. Signal compensation is implemented to remove/attenuate the adverse effects caused by the faults to the system input and output dynamics, leading to a fault-tolerant design.
- iv) Due to stochastic parameter perturbations, the well-known separation theory for observer-based control in determined systems becomes invalid in Itô-type stochastic systems. Apart from unknown input disturbances, the estimation error is also influenced by stochastic perturbations coupled with system states. Hence, the input-to-state stochastic stability principle has to be used to address the fault tolerant control stabilization.
- v) The systems under investigation can be linear, Lipschitz nonlinear, quadratic inner-bounded nonlinear systems corrupted by Brownian parameter perturbations and partially decoupled unknown process disturbances, which can cover a wide class of dynamic processes.

The rest of the paper is organized as follows. Preliminaries are given in Section II. The design of the integrated fault estimation and tolerant control algorithm for linear stochastic systems is addressed in Section III. The technique is then generalized to stochastic Lipschitz nonlinear systems in Section IV. Section V develops robust estimator-based fault tolerant control for stochastic quadratic inner-bounded nonlinear systems which are more general but also more challenging than Lipschitz ones. Simulations on the drive train system of a 4.8MW benchmark

wind turbine, a three-tank system and a numerical nonlinear stochastic system are addressed in Section VI. The paper ends with the conclusion in Section VII.

2. PRELIMINARY

The notations in this paper are standard. The superscript “ T ” represents the transpose of matrices or vectors. \mathcal{R}^n and $\mathcal{R}^{n \times m}$ stand for the n -dimensional Euclidean space and the set of $n \times m$ real matrices, respectively. \mathcal{R}^+ denotes the set of all nonnegative real numbers. $X < 0$ indicates the symmetric matrix X is negative definite, while the notation $X > Y$ means that $X - Y$ is positive definite. I_n denotes the identity matrix with the dimension of $n \times n$, while 0 is a scalar zero or a zero matrix with appropriate zero entries. $|A| = \sqrt{\lambda_{\max}(A^T A)}$; $|x|$ refers to the Euclidean norm of x ; and $|x|_{Tf} = \left(\int_0^{Tf} x^T(\tau) x(\tau) d\tau \right)^{1/2}$. $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq t_0}, \mathcal{P})$ represents a complete probability space with Ω being a sample space, \mathcal{F} being a σ -field, $\{\mathcal{F}_t\}_{t \geq t_0}$ being a filtration and \mathcal{P} being a probability measure. L_∞^m stands for all essentially bounded m -dimensional functions with norm $\|\xi(t)\| = \text{ess. sup.}\{|\xi(t)|, t \geq 0\}$. $\mathbb{E}(\cdot)$ denotes the expectation of a stochastic process, and \forall means for all. For brevity, $\begin{bmatrix} M_1 & M_2 \\ * & M_3 \end{bmatrix} \triangleq \begin{bmatrix} M_1 & M_2 \\ M_2^T & M_3 \end{bmatrix}$.

Definition 1 ([27])

A function $\gamma: \mathcal{R}^+ \rightarrow \mathcal{R}^+$ is said to be a \mathcal{K} -function if it is continuous with $\gamma(0) = 0$, and satisfies:

$$\gamma(\sigma_1) > \gamma(\sigma_2), \forall \sigma_1 > \sigma_2 \geq 0 \quad (1)$$

\mathcal{K}_∞ is the subset of \mathcal{K} -functions that are unbounded.

Definition 2 ([27])

A function $\beta: \mathcal{R}^+ \times \mathcal{R}^+ \rightarrow \mathcal{R}^+$ is said to be a \mathcal{KL} -function if for each fixed $t \geq 0$, the function $\beta(s, t)$ is a \mathcal{K} -function, and for each fixed $s \geq 0$, it decreases to zero as $t \rightarrow \infty$.

Consider the following stochastic system

$$dx(t) = l(t, x(t), v(t))dt + h(t, x(t), v(t))dw(t) \quad (2)$$

where $x(t) \in \mathcal{R}^n$ is system state, with initial value $x(t_0) = x_0 \in \mathcal{R}^n$; $v(t)$ is the input, and $v(t) \in L_\infty^m$; $w(t)$ represents Brownian motion defined on the probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq t_0}, \mathcal{P})$; $l(t, x(t), v(t))$ and $h(t, x(t), v(t))$ stand for system dynamic function and stochastic perturbation distribution function, respectively. Given any function $V(t, x) \in \mathcal{C}^{2 \times 1}\{\mathcal{R}^n \times [t_0, \infty] \rightarrow \mathcal{R}^+\}$, the infinitesimal generator $\mathcal{L}V(t, x)$ is defined as:

$$\mathcal{L}V(t, x) = \frac{\partial V(t, x)}{\partial t} + \frac{\partial V(t, x)}{\partial x} l + \frac{1}{2} \text{trace} \left\{ h^T \frac{\partial^2 V(t, x)}{\partial x^2} h \right\} \quad (3)$$

where $\text{trace} \left\{ h^T \frac{\partial^2 V(t, x)}{\partial x^2} h \right\}$ is called as the Hessian term of \mathcal{L} .

Definition 3 ([28])

System (2) is said to be stochastically input-to-state stable, if $\forall \varepsilon > 0$, there exist functions $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$, such that

$$\mathcal{P}\{|x(t)| \leq \beta(|x_0|, t - t_0) + \gamma(\|v(t)\|)\} \geq 1 - \varepsilon, \forall t \geq t_0, \forall x_0 \in \mathcal{R}^n \setminus \{0\} \quad (4)$$

Remark 1

Since $\gamma(0) = 0$, it can be found that, in zero input situation, stochastically input-to-state stability can necessarily lead to globally asymptotically stability in probability. However, globally asymptotically stability in probability does not imply stochastically input-to-state stability.

Lemma 1 ([28])

System (2) is stochastically input-to-state stable if there exist function V and corresponding \mathcal{K}_∞ functions $\psi_1, \psi_2, \psi_3, \psi_4$ such that

$$(i) \psi_1(|x|) \leq V(t, x) \leq \psi_2(|x|), \forall x \quad (5)$$

$$(ii) \mathcal{L}V(t, x) \leq -\psi_3(|x|) + \psi_4(|v|), \forall x, v \quad (6)$$

3. ROBUST FAULT ESTIMATION AND FAULT TOLERANT CONTROL OF LINEAR STOCHASTIC SYSTEM

Consider the following stochastic linear system in the form of Itô-type stochastic differential equation:

$$\begin{cases} dx(t) = [Ax(t) + Bu(t) + B_d d(t) + B_f f(t)] dt + Wx(t)dw(t) \\ y(t) = Cx(t) + D_f f(t) \end{cases} \quad (7)$$

where $x(t) \in \mathcal{R}^n$ represents the state vector; $u(t) \in \mathcal{R}^m$ stands for the control input vector and $y(t) \in \mathcal{R}^p$ is the measurement output vector; $d(t) \in L_\infty^{l_d}$ is an unknown disturbance vector; $f(t) \in \mathcal{R}^{l_f}$ includes the means of the faults (e.g., actuator faults and/or sensor faults); $w(t)$ is a standard one-dimensional Brownian motion with $\mathbb{E}[w(t)] = 0$ and $\mathbb{E}[w^2(t)] = t$; A, B, C, B_d, B_f, D_f and W are known coefficient matrices with appropriate dimensions. For the simplification of description, in the rest of paper, the time symbol t is omitted.

The means of the faults concerned are assumed to be either incipient or abrupt, which are two typical faults generally existing in practical processes. Therefore, the second-order derivatives of their means should be zero piecewise. For faults whose second order derivatives of their means are not zero but bounded, the bounded signals could be regarded as a part of unknown inputs. Moreover, denote $B_d = [B_{d1} \ B_{d2}]$ and $d = [d_1^T \ d_2^T]^T$. We assume that $d_1 \in \mathcal{R}^{l_{d1}}$ rather than $d_2 \in \mathcal{R}^{l_{d2}}$ can be decoupled.

The aim of this section is to design a robust fault estimation based tolerant controller for system (7). The main objectives include: (i) Estimate full system states and the means of concerned faults simultaneously, and eliminate the influences of the unknown inputs. (ii) Design an observe-based fault tolerant control strategy to guarantee the stochastically input-to-state stability of the closed-loop system, and eliminate the adverse effects from the faults to the system dynamics with the aid of signal compensation.

3.1. Robust fault estimation

This part presents the integration of the augmented system approach and the UIO technique to generate robust estimates of system states and the means of concerned faults simultaneously. The former can establish an auxiliary system vector composed of the original system states and the means of the concerned faults, while the latter is to estimate the auxiliary state vector, and decouple the unknown inputs that can be decoupled.

For system (7), the following augmented system can be constructed

$$\begin{cases} d\bar{x} = (\bar{A}\bar{x} + \bar{B}u + \bar{B}_d d)dt + \bar{W}x dw \\ y = \bar{C}\bar{x} \end{cases} \quad (8)$$

where $\bar{n} = n + 2l_f$, $\bar{x} = [x^T \ \dot{f}^T \ f^T]^T \in \mathcal{R}^{\bar{n}}$, $\bar{A} = \begin{bmatrix} A & 0_{n \times l_f} & B_f \\ 0_{l_f \times n} & 0_{l_f \times l_f} & 0_{l_f \times l_f} \\ 0_{l_f \times n} & I_{l_f} & 0_{l_f \times l_f} \end{bmatrix} \in \mathcal{R}^{\bar{n} \times \bar{n}}$, $\bar{B} = \begin{bmatrix} B^T & 0_{m \times l_f} & 0_{m \times l_f} \end{bmatrix}^T \in \mathcal{R}^{\bar{n} \times m}$, $\bar{B}_d = \begin{bmatrix} B_d^T & 0_{l_d \times l_f} & 0_{l_d \times l_f} \end{bmatrix}^T \in \mathcal{R}^{\bar{n} \times l_d}$, $\bar{W} = \begin{bmatrix} W^T & 0_{n \times l_f} & 0_{n \times l_f} \end{bmatrix}^T \in \mathcal{R}^{\bar{n} \times n}$, $\bar{C} = [C \ 0_{p \times l_f} \ D_f] \in \mathcal{R}^{p \times \bar{n}}$

For the augmented system (8), we design the following unknown input observer:

$$\begin{cases} d\bar{z} = [R\bar{z} + S\bar{B}u + (L_1 + L_2)y]dt \\ \hat{\bar{x}} = \bar{z} + Hy \end{cases} \quad (9)$$

where \bar{z} is the state vector of (9), $\hat{\bar{x}}$ is the estimation of \bar{x} , and R, S, L_1, L_2 and H are all observer gains with appropriate dimensions to be designed.

Let $\bar{e} = \bar{x} - \hat{x}$ which represents the estimation error. In terms of (8) and (9), we have:

$$\begin{aligned}
d\bar{e} &= (I_{\bar{n}} - H\bar{C}) d\bar{x} - d\bar{z} \\
&= \{(I_{\bar{n}} - H\bar{C}) (\bar{A}\bar{x} + \bar{B}u + \bar{B}_d d) - R\bar{z} - S\bar{B}u - (L_1 + L_2)y\}dt \\
&\quad + (I_{\bar{n}} - H\bar{C})\bar{W}x dw \\
&= \{(I_{\bar{n}} - H\bar{C})\bar{A}\bar{x} - L_1\bar{C}\bar{x} + (I_{\bar{n}} - H\bar{C})\bar{B}u + (I_{\bar{n}} - H\bar{C})\bar{B}_d d - R\bar{z} - S\bar{B}u \\
&\quad - L_2y\}dt + (I_{\bar{n}} - H\bar{C})\bar{W}x dw \\
&= \{[(I_{\bar{n}} - H\bar{C})\bar{A} - L_1\bar{C}]\bar{x} - R\hat{x} + [(I_{\bar{n}} - H\bar{C}) - S]\bar{B}u + (I_{\bar{n}} - H\bar{C})\bar{B}_{d1}d_1 \\
&\quad + (I_{\bar{n}} - H\bar{C})\bar{B}_{d2}d_2 + (RH - L_2)y\}dt + (I_{\bar{n}} - H\bar{C})\bar{W}x dw
\end{aligned} \tag{10}$$

If the observer gains satisfy the following conditions:

$$(I_{\bar{n}} - H\bar{C})\bar{B}_{d1} = 0 \tag{11}$$

$$R = \bar{A} - H\bar{C}\bar{A} - L_1\bar{C} \tag{12}$$

$$S = I_{\bar{n}} - H\bar{C} \tag{13}$$

$$L_2 = RH \tag{14}$$

the state estimation error can be simplified as

$$d\bar{e} = (R\bar{e} + S\bar{B}_{d2}d_2)dt + S\bar{W}x dw \tag{15}$$

In order to make (11)-(14) solvable, we have the following assumptions:

Assumption 1

$$\text{rank}(CB_{d1}) = \text{rank}(B_{d1}).$$

Assumption 2

$$\begin{bmatrix} A & B_f & B_{d1} \\ C & D_f & 0 \end{bmatrix} \text{ is of full column rank.}$$

Assumption 3

$$\text{rank} \begin{bmatrix} sI_n - A & B_{d1} \\ C & 0 \end{bmatrix} = n + l_{d1}$$

for all non-zero s with $\text{Re}(s) \geq 0$.

According to [22], the above assumptions are to ensure that (11) can be solved, and observer of the augmented system exists. Moreover, a special solution of (11) is

$$H^* = \bar{B}_{d1}[(\bar{C}\bar{B}_{d1})^T(\bar{C}\bar{B}_{d1})]^{-1}(\bar{C}\bar{B}_{d1})^T \tag{16}$$

By deriving H from (16) to satisfy condition (11), d_1 can be decoupled. However, the unknown input d_2 cannot be decoupled, and still exists in the error dynamic. It is evident that additional optimization approach should be employed to determine other observer gains so that the influence of d_2 can be attenuated. Furthermore, it is noticed that the stochastic perturbation term $S\bar{W}xdw$ exists in the error dynamic equation (15), therefore, the performance of the estimator depends not only on appropriate observer gains, but also on controlled states. As a result, before we choose the observer gains, a proper controller should be taken into account.

3.2. Robust estimation-based fault tolerant control

As stated in the aforementioned part, the estimation error dynamics rely on the design of observer gains and the controlled states. Therefore, observer-based controller should be designed as a whole. Now let us move on to deal with the observer-based fault tolerant control method.

Consider the following control law

$$u = \bar{K}\hat{x} = [K \quad 0 \quad K_f] \begin{bmatrix} \hat{x} \\ \hat{f} \\ \hat{f} \end{bmatrix} = K\hat{x} + K_f\hat{f} \quad (17)$$

where K and K_f are control gains to be determined, \hat{x} , \hat{f} and \hat{f} represent the estimates of x , \dot{f} and f respectively. Moreover, K should be selected to guarantee the convergence of the closed-loop system, while K_f is designed to compensate the influences of the faults.

Based on the estimation of \hat{x} , the estimates of the original system state and fault vector can be reconstructed as

$$\hat{x} = [I_n \quad 0_{n \times l_f} \quad 0_{n \times l_f}] \hat{\hat{x}} \quad (18)$$

and

$$\hat{f} = [0_{l_f \times n} \quad 0_{l_f \times l_f} \quad I_{l_f}] \hat{\hat{x}} \quad (19)$$

Suppose

$$\text{rank}[B \quad B_f] = \text{rank } B \quad (20)$$

and choose

$$K_f = -B^+B_f \quad (21)$$

Therefore, we have

$$B_f + BK_f = 0 \quad (22)$$

Substituting (17) into dynamic equation of system (7), it follows that

$$\begin{aligned} dx &= (Ax + Bu + B_d d + B_f f) dt + Wx dw \\ &= (Ax + BK\hat{x} + BK_f\hat{f} + B_d d + B_f f) dt + Wx dw \\ &= (Ax + BKx - BKJ_0\bar{e} + BK_f\hat{f} + B_d d + B_f f - B_f\hat{f} + B_f\hat{f}) dt + Wx dw \\ &= [(A + BK)x - BKJ_0\bar{e} + BK_f\hat{f} + B_d d + B_f J_2\bar{e} + B_f\hat{f}] dt + Wx dw \\ &= [(A + BK)x - BKJ_0\bar{e} + (B_f + BK_f)\hat{f} + B_d d + B_f J_2\bar{e}] dt + Wx dw \\ &= [(A + BK)x - BKJ_0\bar{e} + B_d d + B_f J_2\bar{e}] dt + Wx dw \\ &= [(A + BK)x - B_e\bar{e} + B_d d] dt + Wx dw \end{aligned} \quad (23)$$

where $J_0 = [I_n \ 0_{n \times l_f} \ 0_{n \times l_f}]$, $J_2 = [0_{l_f \times n} \ 0_{l_f \times l_f} \ I_{l_f}]$, and $B_e = BKJ_0 - B_f J_2$.

Using $-D_f\hat{f}$ to compensate the measurement output, we have

$$y_c = y - D_f J_2 \hat{x} = Cx + D_f f - D_f \hat{f} = Cx + D_f J_2 \bar{e} \quad (24)$$

From (23) and (24), the following closed-loop system can be established

$$\begin{cases} dx = [(A + BK)x - B_e\bar{e} + B_d d] dt + Wx dw \\ d\bar{e} = (R\bar{e} + S\bar{B}_{d2}d_2) dt + S\bar{W}x dw \\ y_c = Cx + D_e\bar{e} \end{cases} \quad (25)$$

where $D_e = D_f J_2$.

Then we are moving on to design the observer and controller gains to make system (25) stochastically input-to-state stable, and satisfy the robust performance index:

$$\mathbb{E}(|y_c|_{T_f}^2) < \gamma_1^2 \mathbb{E}(|d_1|_{T_f}^2) + \gamma_2^2 \mathbb{E}(|d_2|_{T_f}^2) \quad (26)$$

It is noticed that both the system dynamics and error dynamics are subjected to state stochastic fluctuation, which makes it challenging to design observer and controller gains simultaneously. In order to simplify the challenging matrix problem, we firstly design control gain K , such that $A + BK$ is a Hurwitz matrix, then choose a proper observer gain L_1 to guarantee the stochastically input-to-state stability of closed-loop plant (25) with robust performance index (26). Furthermore, in the case of observer-based fault tolerant control, the design of observer gain L_1 should make the estimation error dynamics reach the steady states faster than the control system dynamics. Therefore, before we determine the observer gain L_1 , the following lemma is introduced:

Lemma 2 ([29])

Consider the vertical strip defined by $\mathcal{D}(a) = \{x + jy \in \mathcal{C} : x < -a, a > 0\}$, a matrix A has all its eigenvalues in $\mathcal{D}(a)$ if and only if there exists a positive definite matrix X , such that

$$A^T X + XA + 2aX < 0 \quad (27)$$

Therefore, based on a designed K , the following theorem is proposed to design L_1 .

Theorem 1

For system (7), there exist an unknown input observer in the form of (9), and the tolerant control laws in the forms of (17) and (24), such that the closed-loop system (25) is stochastically input-to-state stable satisfying the robust performance index (26), if there exist positive definite matrices P , \bar{P} , Q , and \bar{Q} , and matrix Y , such that

$$\begin{bmatrix} \Omega_{11} & -PB_e + C^T D_e & PB_{d1} & PB_{d2} \\ * & \Omega_{22} & 0 & \bar{P}S\bar{B}_{d2} \\ * & * & -\gamma_1^2 I_{l_{d1}} & 0 \\ * & * & * & -\gamma_2^2 I_{l_{d2}} \end{bmatrix} < 0 \quad (28)$$

and

$$\bar{P}S\bar{A} + \bar{A}^T S^T \bar{P} - Y\bar{C} - \bar{C}^T Y^T + 2a\bar{P} < 0 \quad (29)$$

where

$$\Omega_{11} = P(A + BK) + (A + BK)^T P + W^T P W + \bar{W}^T S^T \bar{P} S \bar{W} + Q + C^T C,$$

$$\Omega_{22} = \bar{P}S\bar{A} + \bar{A}^T S^T \bar{P} - Y\bar{C} - \bar{C}^T Y^T + \bar{Q} + D_e^T D_e,$$

$$Y = \bar{P}L_1,$$

$$a = \beta\theta_l, \theta_l = -\min\{\text{Re}[\lambda_i(A + BK)]\} > 0, i = \{1, 2, \dots, n\} \text{ and } \beta > 1.$$

We can thus calculate $L_1 = \bar{P}^{-1}Y$.

Proof

According to *Lemma 1*, in order to prove the stability, we should establish a Lyapunov function satisfying (5) and (6). Here, we choose the candidate as $V = V_1 + V_2$, where $V_1 = x^T P x$ and $V_2 = \bar{e}^T P \bar{e}$. We can notice that

$$V = \tilde{x}^T \tilde{P} \tilde{x} \quad (30)$$

where $\tilde{x} = [x^T \quad \bar{e}^T]^T$, $\tilde{P} = \begin{bmatrix} P & 0 \\ 0 & \bar{P} \end{bmatrix}$. Then we can find

$$\lambda_{\min}(\tilde{P})|\tilde{x}|^2 \leq V \leq \lambda_{\max}(\tilde{P})|\tilde{x}|^2 \quad (31)$$

which implies V satisfies (5) with $\psi_1 = \lambda_{\min}(\tilde{P})|\tilde{x}|^2$, $\psi_2 = \lambda_{\max}(\tilde{P})|\tilde{x}|^2$ in *Lemma 1*. Taking infinitesimal generator (3) along the state trajectories of (25), by using Itô formula, we have

$$\mathcal{L}V_1 = x^T [P(A + BK) + (A + BK)^T P]x - 2x^T P B_e \bar{e} + 2x^T P B_d d + x^T W^T P W x \quad (32)$$

and

$$\mathcal{L}V_2 = \bar{e}^T (\bar{P}R + R^T \bar{P})\bar{e} + 2\bar{e}^T \bar{P}S\bar{B}_{d2}d_2 + x^T \bar{W}^T S^T \bar{P}S\bar{W}x \quad (33)$$

Therefore, we have

$$\begin{aligned} \mathcal{L}V &= \mathcal{L}V_1 + \mathcal{L}V_2 \\ &= x^T [P(A + BK) + (A + BK)^T P + W^T P W + \bar{W}^T S^T \bar{P}S\bar{W}]x - 2x^T P B_e \bar{e} \\ &\quad + 2x^T P B_d d + \bar{e}^T (\bar{P}R + R^T \bar{P})\bar{e} + 2\bar{e}^T \bar{P}S\bar{B}_{d2}d_2 \end{aligned} \quad (34)$$

Adding and subtracting $x^T Q x + \bar{e}^T \bar{Q} \bar{e} - \gamma_1^2 d_1^T d_1 - \gamma_2^2 d_2^T d_2$ to $\mathcal{L}V$, we can obtain

$$\mathcal{L}V = \begin{bmatrix} x^T & \bar{e}^T & d_1^T & d_2^T \end{bmatrix} \Psi \begin{bmatrix} x \\ \bar{e} \\ d_1 \\ d_2 \end{bmatrix} - x^T Q x - \bar{e}^T \bar{Q} \bar{e} + \gamma_1^2 d_1^T d_1 + \gamma_2^2 d_2^T d_2 \quad (35)$$

where

$$\Psi = \begin{bmatrix} \Psi_{11} & -P B_e & P B_{d1} & P B_{d2} \\ * & \Psi_{22} & 0 & \bar{P}S\bar{B}_{d2} \\ * & * & -\gamma_1^2 I_{l_{d1}} & 0 \\ * & * & * & -\gamma_2^2 I_{l_{d2}} \end{bmatrix} \quad (36)$$

$$\Psi_{11} = P(A + BK) + (A + BK)^T P + W^T P W + \bar{W}^T S^T \bar{P}S\bar{W} + Q,$$

$$\Psi_{22} = \bar{P}S\bar{A} + \bar{A}^T S^T \bar{P} - \bar{P}L_1\bar{C} - \bar{C}^T L_1^T \bar{P}^T + \bar{Q}.$$

From the LMI (28), one has

$$\Psi < 0 \quad (37)$$

which indicates

$$\begin{aligned} \mathcal{L}V &\leq -x^T Q x - \bar{e}^T \bar{Q} \bar{e} + \gamma_1^2 d_1^T d_1 + \gamma_2^2 d_2^T d_2 \\ &= -\begin{bmatrix} x^T & \bar{e}^T \end{bmatrix} \begin{bmatrix} Q & 0 \\ 0 & \bar{Q} \end{bmatrix} \begin{bmatrix} x \\ \bar{e} \end{bmatrix} + \gamma_1^2 d_1^T d_1 + \gamma_2^2 d_2^T d_2 \end{aligned} \quad (38)$$

Since Q and \bar{Q} are both positive definite, we have

$$\tilde{Q} = \begin{bmatrix} Q & 0 \\ 0 & \bar{Q} \end{bmatrix} > 0 \quad (39)$$

indicating we can find a scalar $\bar{\lambda} > 0$ such that

$$\mathcal{L}V \leq -\bar{\lambda} |\tilde{x}|^2 + \gamma_1^2 |d_1|^2 + \gamma_2^2 |d_2|^2 \quad (40)$$

As a result, we can conclude the closed-loop system (25) is stochastically input-to-state sable with

$$\psi_3(\tilde{x}) = \bar{\lambda} |\tilde{x}|^2 \quad (41)$$

and

$$\psi_4(|d|) = \gamma_1^2 |d_1|^2 + \gamma_2^2 |d_2|^2 \quad (42)$$

Now we move on to discuss the robustness of the observer-based fault tolerant control. Consider the following performance index:

$$\begin{aligned} \Gamma &= \mathbb{E} \left\{ \int_0^{Tf} [y_c^T(\tau) y_c(\tau) - \gamma_1^2 d_1^T(\tau) d_1(\tau) - \gamma_2^2 d_2^T(\tau) d_2(\tau)] d\tau \right\} \\ &= \mathbb{E} \left\{ \int_0^{Tf} [x^T(\tau) C^T C x(\tau) + \bar{e}^T(\tau) D_e^T D_e \bar{e}(\tau) + 2x^T(\tau) C^T D_e \bar{e}(\tau) \right. \\ &\quad \left. - \gamma_1^2 d_1^T(\tau) d_1(\tau) - \gamma_2^2 d_2^T(\tau) d_2(\tau)] d\tau \right\} \end{aligned} \quad (43)$$

It is obvious that condition (26) is equivalent to the condition $\Gamma < 0$. Then, adding and subtracting $\mathbb{E}(\int_0^t \mathcal{L}V d\tau)$ to Γ , and using $Y = \bar{P}L_1$, one has:

$$\Gamma = \mathbb{E} \left\{ \int_0^{Tf} [x^T \quad \bar{e}^T \quad d_1^T \quad d_2^T] \Omega \begin{bmatrix} x \\ \bar{e} \\ d_1 \\ d_2 \end{bmatrix} - x^T(\tau) Q x(\tau) - \bar{e}^T(\tau) \bar{Q} \bar{e}(\tau) \right\} - \mathbb{E} \left(\int_0^{Tf} \mathcal{L}V d\tau \right) \quad (44)$$

where

$$\Omega = \begin{bmatrix} \Omega_{11} & -PB_e + C^T D_e & PB_{d1} & PB_{d2} \\ * & \Omega_{22} & 0 & \bar{P}S\bar{B}_{d2} \\ * & * & -\gamma_1^2 I_{l_{d1}} & 0 \\ * & * & * & -\gamma_2^2 I_{l_{d2}} \end{bmatrix}$$

$$\Omega_{11} = P(A + BK) + (A + BK)^T P + W^T P W + \bar{W}^T S^T \bar{P} S \bar{W} + Q + C^T C,$$

$$\Omega_{22} = \bar{P}S\bar{A} + \bar{A}^T S^T \bar{P} - Y\bar{C} - \bar{C}^T Y^T + \bar{Q} + D_e^T D_e.$$

It is not hard to find

$$\mathbb{E}(\int_0^{T_f} \mathcal{L}V d\tau) = \mathbb{E}(V) > 0 \quad (45)$$

One can have $\Omega < 0$ from the LMI (28), thus one can derive $\Gamma < 0$, which indicates the (26) can be satisfied.

From *Lemma 2*, the LMI (29) implies

$$\text{Re}[\lambda_i(R)] < -a, \quad i = \{1, 2, \dots, \bar{n}\} \quad (46)$$

Noticing that $a = \beta\theta_l$, where $\theta_l = -\min\{\text{Re}[\lambda_i(A + BK)]\} > 0$ and $\beta > 1$, one can know the response of the estimation error dynamics is faster than the system dynamics. This completes the proof.

Remark 2

As aforementioned, control gain K should be designed to make $A + BK$ Hurwitz, which means the eigenvalues of the matrix $A + BK$ are located on the left half complex plane. For some practical applications, it can be required that the eigenvalues of the matrix $A + BK$ are settled in a specific region $\mathcal{D}(c, \mu, \delta) = \{x + jy \in \mathcal{C} : x < -c, |x + jy| < \mu, \tan(\delta)x < -|y|\}$, where c, μ and δ are positive scalars, which is to ensure a minimum decay rate c , a minimum damping ratio $\varsigma = \cos(\delta)$, and a maximum un-damped natural frequency $\omega_d = \mu\sin(\delta)$. According to [29], we can derive that if there exist a positive definite matrix X and matrix Z such that

$$AX + BZ + XA^T + Z^T B^T + 2cX < 0 \quad (47)$$

$$\begin{bmatrix} -\mu X & AX + BZ \\ * & -\mu X \end{bmatrix} < 0 \quad (48)$$

$$\begin{bmatrix} \Delta_{11} & \Delta_{12} \\ \Delta_{21} & \Delta_{22} \end{bmatrix} < 0 \quad (49)$$

where $Z = KX$, $\Delta_{11} = \sin(\delta)(AX + BZ + XA^T + Z^T B^T)$, $\Delta_{12} = \cos(\delta)(AX + BZ - XA^T - Z^T B^T)$, $\Delta_{21} = \cos(\delta)(XA^T + Z^T B^T - AX - BZ)$, and $\Delta_{22} = \sin(\delta)(AX + BZ + XA^T + Z^T B^T)$, then $\lambda_i(A + BK) \in \mathcal{D}(c, \mu, \delta), \forall i = \{1, 2, \dots, n\}$. After obtaining X and Z , the control gain K can be calculated as $K = ZX^{-1}$ to constrain the poles of $A + BK$ to lie in a prescribed stable region $\mathcal{D}(c, \mu, \delta)$. This design bounds the maximum overshoot, the frequency of oscillatory modes, the delay time, rise time, and the settling time.

Remark 3

LMI algorithms are utilized to solve observer gains and controller gains. It is noted that the design is carried out off line, therefore, the computation complexity has not direct impact on the

real-time implementation of the proposed diagnosis and tolerant control algorithms. In other words, the commutation complexity discussion is out of the concern of this paper.

3.3. Design procedures of robust estimation-based fault tolerant control for stochastic linear systems

Now, it is time to conclude the design procedure of the robust fault estimation and fault tolerant control strategies.

Procedure 1 (Fault tolerant control algorithm by integrating state/fault estimation and signal compensation for stochastic linear systems)

- i) Construct the augmented system in the form of (8) for system (7).
- ii) Select the matrix H^* in the form of (16), and S can be calculated in terms of $S = I_{\bar{n}} - H\bar{C}$.
- iii) Design control gain K to make $A + BK$ Hurwitz. For a certain desired region $\mathcal{D}(c, \mu, \delta)$, solve LMIs (47)-(49) to determine the control gain K such that all poles of $A + BK$ are settled in $\mathcal{D}(c, \mu, \delta)$. Denote $\theta_l = -\min\{\text{Re}[\lambda_i(A + BK)]\}$, and $a = \beta\theta_l$. β can be chosen between 2 and 5 such that the response of the estimation error is reasonably faster than that of the system dynamics.
- iv) Solve the LMIs (28) and (29) to obtain P , Q , \bar{P} , \bar{Q} and matrix Y . The observer gain is thus calculated as $L_1 = \bar{P}^{-1}Y$.
- v) Calculate the other observer gains R and L_2 following the formulas (11) and (14), respectively.
- vi) Implement the robust unknown input observer (9) to produce the augmented estimate $\hat{\bar{x}}$, leading to simultaneous estimates of the system state and mean of fault (\hat{x} and \hat{f}) in the forms of (18) and (19), respectively.
- vii) Implement the tolerant control law $u = \bar{K}\hat{\bar{x}}$ and $y_c = y - D_f\hat{f}$, where $\bar{K} = [K \quad 0 \quad K_f]$ and $K_f = -B^+B_f$.

4. ROBUST FAULT ESTIMATION AND FAULT TOLERANT CONTROL OF LIPSCHITZ NONLINEAR STOCHASTIC SYSTEM

In the last section, a robust fault tolerant control technique has been developed for stochastic linear systems. It is well-known nonlinear properties widely exist in many practical dynamics, which motivates us to extend the approach proposed in Section III to nonlinear systems. In this section, robust fault estimation-based fault tolerant control approach is proposed for Lipschitz

nonlinear systems subjected to unexpected faults, unknown inputs and stochastic parameter perturbations. The considered stochastic nonlinear systems can be represented by the following Itô-type differential equations:

$$\begin{cases} dx = [Ax + Bu + B_d d + B_f f + \phi(t, x, u)]dt + Wx dw \\ y = Cx + D_f f \end{cases} \quad (50)$$

where $\phi(t, x, u) \in \mathcal{R}^n$ is a real nonlinear vector function with Lipschitz constant θ , namely,

$$\phi(t, x, u) - \phi(t, \hat{x}, u) \leq \theta(|x - \hat{x}|), \forall (t, x, u), (t, \hat{x}, u) \in \mathcal{R} \times \mathcal{R}^n \times \mathcal{R}^m \quad (51)$$

and the other symbols are the same as defined as system (7).

In order to estimate system states and the means of faults at the same time, an auxiliary system is constructed as follows, by considering the faults as augmented system states:

$$\begin{cases} d\bar{x} = [\bar{A}\bar{x} + \bar{B}u + \bar{B}_d d + \bar{\phi}(t, x, u)]dt + \bar{W}x dw \\ y = \bar{C}\bar{x} \end{cases} \quad (52)$$

where

$$\bar{\phi}(t, x, u) = [\phi^T(t, x, u) \quad 0_{1 \times l_f} \quad 0_{1 \times l_f}]^T \in \mathcal{R}^{\bar{n}} \quad (53)$$

and the other symbols are defined the same as those in system (8). In the rest of this paper, the symbols are of the same meanings with those defined in section III if not stated specifically.

For system (52), the following unknown input observer is constructed:

$$\begin{cases} d\bar{z} = [R\bar{z} + S\bar{B}u + (L_1 + L_2)y + S\bar{\phi}(t, \hat{x}, u)]dt \\ \hat{\bar{x}} = \bar{z} + Hy \end{cases} \quad (54)$$

Defining the estimation error to be $\bar{e} = \bar{x} - \hat{\bar{x}}$ and $\tilde{\phi} = \bar{\phi}(t, x, u) - \bar{\phi}(t, \hat{x}, u)$. Then the following error dynamics can be obtained if (11)-(14) are satisfied

$$d\bar{e} = (R\bar{e} + S\bar{B}_{d2}d_2 + S\tilde{\phi})dt + S\bar{W}x dw \quad (55)$$

Under the designed observer-based fault tolerant controller (17) and (24), the overall closed-loop system can be obtained as follows:

$$\begin{cases} dx = [(A + BK)x - B_e \bar{e} + B_d d + \phi(t, x, u)]dt + Wx dw \\ d\bar{e} = (R\bar{e} + S\bar{B}_{d2}d_2 + S\tilde{\phi})dt + S\bar{W}x dw \\ y_c = Cx + D_e \bar{e} \end{cases} \quad (56)$$

It is obvious that both actuator and sensor faults can be well compensated as long as the estimation is good enough. Based on Section III, the control gain K can be selected to make $A + BK$ Hurwitz. To meet some instantaneous response requirements, the eigenvalue of $A + BK$ can

be located within certain region by solving LMIs (47)-(49). Therefore, based on the designed K , we aim to determine proper observer gains such that: (i) the overall closed-loop system (56) is stochastically input-to-state stable; (ii) condition (26) is satisfied, which means the controlled output is robust against unknown perturbations.

It can be noticed that the existence of nonlinear item $\phi(t, x, u)$ and $\tilde{\phi}$ in plant (56) make the linear method to design observer gain not applicable. In consequence, when we design observer gain L_1 additional techniques should be employed to deal with the nonlinearities.

Lemma 3 ([30])

For any matrices $M \in \mathcal{R}^{s \times t}$, $N \in \mathcal{R}^{t \times s}$, a time-varying matrix $F(t) \in \mathcal{R}^{t \times t}$ with $\|F(t)\| \leq 1$ and any scalar $\varepsilon > 0$, we have:

$$MF(t)N + N^T F^T(t)M^T \leq \varepsilon^{-1}MM^T + \varepsilon N^T N \quad (57)$$

Lemma 4 (Schur complement, [31])

Let $S = \begin{bmatrix} S_{11} & S_{12} \\ * & S_{22} \end{bmatrix}$ be a symmetric matrix. $S < 0$ is equivalent to $S_{22} < 0$ and $S_{11} - S_{12}S_{22}^{-1}S_{12}^T < 0$.

Theorem 2

For system (50), there exist an unknown input observer in the form of (54), and the tolerant control laws in the forms of (17) and (24), such that the closed-loop system (56) is stochastically input-to-state stable satisfying the robust performance index (26), if there exist positive definite matrices P , \bar{P} , Q , and \bar{Q} , and matrix Y , such that

$$\begin{bmatrix} \Xi_{11} & -PB_e + C^T D_e & PB_{d1} & PB_{d2} & P & 0 \\ * & \Xi_{22} & 0 & \bar{P}S\bar{B}_{d2} & 0 & \bar{P}S \\ * & * & -\gamma_1^2 I_{l_{d1}} & 0 & 0 & 0 \\ * & * & * & -\gamma_2^2 I_{l_{d2}} & 0 & 0 \\ * & * & * & * & -\varepsilon_1 I_n & 0 \\ * & * & * & * & * & -\varepsilon_2 I_{\bar{n}} \end{bmatrix} < 0 \quad (58)$$

and

$$\bar{P}S\bar{A} + \bar{A}^T S^T \bar{P} - Y\bar{C} - \bar{C}^T Y^T + 2a\bar{P} < 0 \quad (59)$$

where

$$\Xi_{11} = P(A + BK) + (A + BK)^T P + \varepsilon_1 \theta^2 + W^T P W + \bar{W}^T S^T \bar{P} S \bar{W} + C^T C + Q,$$

$$\Xi_{22} = \bar{P}S\bar{A} + \bar{A}^T S^T \bar{P} - Y\bar{C} - \bar{C}^T Y^T + D_e^T D_e + \varepsilon_2 \theta^2 I_{\bar{n}} + \bar{Q},$$

$$Y = \bar{P}L_1,$$

ε_1 and ε_2 are given positive scalars,

$a = \beta\theta_l, \theta_l = -\min\{\text{Re}[\lambda_i(A+BK)]\} > 0, i = \{1, 2, \dots, n\}$ and $\beta > 1$.

One can thus calculate $L_1 = \bar{P}^{-1}Y$.

Proof

Choosing the Lyapunov function in the form of (30), and similarly to the proof of *Theorem 1*, we can know it satisfies condition (5) in *Lemma 1*. Taking infinitesimal generator along the state trajectories of (56), by using Itô formula, it follows that:

$$\begin{aligned} \mathcal{L}V &= \mathcal{L}V_1 + \mathcal{L}V_2 \\ &= x^T [P(A+BK) + (A+BK)^T P]x - 2x^T P B_e \bar{e} + 2x^T P B_d d + 2x^T P \phi(t, x, u) \\ &\quad + x^T W^T P W x + \bar{e}^T (\bar{P}R + R^T \bar{P}) \bar{e} + 2\bar{e}^T \bar{P} S \bar{B}_{d2} d_2 + 2\bar{e}^T \bar{P} S \tilde{\phi} + x^T \bar{W}^T S^T \bar{P} S \bar{W} x \end{aligned} \quad (60)$$

According to *Lemma 3*, we have

$$\begin{aligned} \mathcal{L}V &\leq x^T [P(A+BK) + (A+BK)^T P + \varepsilon_1 \theta^2 I_n + \varepsilon_1^{-1} P P]x \\ &\quad - 2x^T P B_e \bar{e} + x^T W^T P W x + 2x^T P B_d d \\ &\quad + \bar{e}^T (\bar{P}R + R^T \bar{P} + \varepsilon_2 \theta^2 I_{\bar{n}} + \varepsilon_2^{-1} \bar{P} S S^T \bar{P}) \bar{e} + 2\bar{e}^T \bar{P} S \bar{B}_{d2} d_2 + x^T \bar{W}^T S^T \bar{P} S \bar{W} x \\ &= x^T [P(A+BK) + (A+BK)^T P + \varepsilon_1 \theta^2 I_n + \varepsilon_1^{-1} P P + Q \\ &\quad + W^T P W + \bar{W}^T S^T \bar{P} S \bar{W}]x - 2x^T P B_e \bar{e} \\ &\quad + \bar{e}^T (\bar{P}R + R^T \bar{P} + \varepsilon_2 \theta^2 I_{\bar{n}} + \varepsilon_2^{-1} \bar{P} S S^T \bar{P} + \bar{Q}) \bar{e} \\ &\quad + 2x^T P B_d d + 2\bar{e}^T \bar{P} S \bar{B}_{d2} d_2 - x^T Q x - \bar{e}^T \bar{Q} \bar{e} \} \\ &= [x^T \quad \bar{e}^T \quad d_1^T \quad d_2^T] \Pi \begin{bmatrix} x \\ \bar{e} \\ d_1 \\ d_2 \end{bmatrix} - x^T Q x - \bar{e}^T \bar{Q} \bar{e} + \gamma_1^2 d_1^T d_1 + \gamma_2^2 d_2^T d_2 \end{aligned} \quad (61)$$

where

$$\Pi = \begin{bmatrix} \Pi_{11} & -P B_e & P B_{d1} & P B_{d2} \\ * & \Pi_{22} & 0 & \bar{P} S \bar{B}_{d2} \\ * & * & -\gamma_1^2 I_{l_{d1}} & 0 \\ * & * & * & -\gamma_2^2 I_{l_{d2}} \end{bmatrix}$$

$\Pi_{11} = P(A + BK) + (A + BK)^T P + \varepsilon_1 \theta^2 I_n + \varepsilon_1^{-1} PP + W^T P W + \bar{W}^T S^T \bar{P} S \bar{W} + Q$, $\Pi_{22} = \bar{P} S \bar{A} + \bar{A}^T S^T \bar{P} - \bar{P} L_1 \bar{C} - \bar{C}^T L_1^T \bar{P} + \varepsilon_2 \theta^2 I_{\bar{n}} + \varepsilon_2^{-1} \bar{P} S S^T \bar{P} + \bar{Q}$, ε_1 and ε_2 are given positive scalars.

From LMI (58), we can see $\Pi < 0$, indicating

$$\begin{aligned} \mathcal{L}V &\leq -x^T Q x - \bar{e}^T \bar{Q} \bar{e} + \gamma_1^2 d_1^T d_1 + \gamma_2^2 d_2^T d_2 \\ &= -\tilde{x}^T \tilde{Q} \tilde{x} + \gamma_1^2 d_1^T d_1 + \gamma_2^2 d_2^T d_2 \end{aligned} \quad (62)$$

A positive scalar $\bar{\lambda}$ can be found such that

$$\mathcal{L}V \leq -\bar{\lambda} |\tilde{x}|^2 + \gamma_1^2 |d_1|^2 + \gamma_2^2 |d_2|^2 \quad (63)$$

According to *Lemma 1*, system (56) is stochastically input-to-state stable with $\psi_3(\tilde{x}) = \bar{\lambda} |\tilde{x}|^2$ and $\psi_4(|d|) = \gamma_1^2 |d_1|^2 + \gamma_2^2 |d_2|^2$.

In terms of the robustness of tolerant control, we can calculate that

$$\begin{aligned} \Gamma &= \mathbb{E} \left\{ \int_0^{Tf} [y_c^T(\tau) y_c(\tau) - \gamma_1^2 d_1^T(\tau) d_1(\tau) - \gamma_2^2 d_2^T(\tau) d_2(\tau)] d\tau \right\} \\ &\leq \mathbb{E} \left\{ \int_0^{Tf} \begin{bmatrix} x^T & \bar{e}^T & d_1^T & d_2^T \end{bmatrix} \Lambda \begin{bmatrix} x \\ \bar{e} \\ d_1 \\ d_2 \end{bmatrix} d\tau \right\} - \mathbb{E} \left(\int_0^{Tf} \mathcal{L}V d\tau \right) \end{aligned} \quad (64)$$

where

$$\Lambda = \begin{bmatrix} \Lambda_{11} & -PB_e + C^T D_e & PB_{d1} & PB_{d2} \\ * & \Lambda_{22} & 0 & \bar{P} S \bar{B}_{d2} \\ * & * & -\gamma_1^2 I_{l_{d1}} & 0 \\ * & * & * & -\gamma_2^2 I_{l_{d2}} \end{bmatrix}$$

$$\Lambda_{11} = P(A + BK) + (A + BK)^T P + \varepsilon_1 \theta^2 I_n + \varepsilon_1^{-1} PP + W^T P W + \bar{W}^T S^T \bar{P} S \bar{W} + C^T C + Q,$$

$$\Lambda_{22} = \bar{P} S \bar{A} + \bar{A}^T S^T \bar{P} - \bar{P} L_1 \bar{C} - \bar{C}^T L_1^T \bar{P} + D_e^T D_e + \varepsilon_2 \theta^2 I_{\bar{n}} + \varepsilon_2^{-1} \bar{P} S S^T \bar{P} + \bar{Q}.$$

Since $\mathbb{E} \left(\int_0^{Tf} \mathcal{L}V d\tau \right) > 0$, if $\Lambda < 0$, we can get $\Gamma < 0$, leading to $\mathbb{E}(|y_c|_{Tf}^2) < \gamma_1^2 \mathbb{E}(|d_1|_{Tf}^2) + \gamma_2^2 \mathbb{E}(|d_2|_{Tf}^2)$. Nevertheless, it is noted that $\Lambda < 0$ is not linear and difficult to be solved. According to *Lemma 4*, and using $Y = \bar{P} L_1$, $\Lambda < 0$ is equivalent with LMI (58). As a result, LMI (58) can guarantee the stochastically input-to-state stability of system (56) and robust fault tolerant control requirement (26).

Similar to *Theorem 1*, condition (59) is to guarantee that the convergence speed of estimation error dynamics is faster than that of the control system dynamics. This completes the proof.

Now, we can conclude the design procedure of the robust fault estimation and fault tolerant control strategies for stochastic Lipschitz nonlinear systems.

Procedure 2 (Fault tolerant control algorithm by integrating state/fault estimation and signal compensation for stochastic Lipschitz nonlinear systems)

- i) Construct the augmented system in the form of (52) for system (50).
- ii) Select observer gains H and S following step ii) in *Procedure 1*.
- iii) Design control gain K in the same way with step iii) in *Procedure 1*.
- iv) Solve the LMIs (58) and (59) to obtain P, Q, \bar{P}, \bar{Q} and matrix Y . The observer gain is thus calculated as $L_1 = \bar{P}^{-1}Y$.
- v) Calculate the other observer gains R and L_2 following the formulas (11) and (14), respectively.
- vi) Implement the robust unknown input observer (54) to produce the augmented estimate \hat{x} , leading to simultaneous estimates of the system state and mean of fault (\hat{x} and \hat{f}) in the forms of (18) and (19), respectively.
- vii) Implement the tolerant control law $u = \bar{K}\hat{x}$ and $y_c = y - D_f\hat{f}$, where $\bar{K} = [K \quad 0 \quad K_f]$ and $K_f = -B^+B_f$.

5. ROBUST FAULT ESTIMATION AND FAULT TOLERANT CONTROL OF QUADRATIC INNER-BOUNDED NONLINEAR STOCHASTIC SYSTEM

In Section IV, we consider robust fault tolerant control for Lipschitz stochastic nonlinear systems. In some real plants, the nonlinear items cannot satisfy Lipschitz condition. In this section, we consider robust fault tolerant control for quadratic inner-bounded stochastic nonlinear systems, which describe a more general case than Lipschitz nonlinear ones. The systems under consideration can be represented in the following form:

$$\begin{cases} dx = [Ax + Bu + B_d d + B_f f + g(t, x, u)] dt + Wx dw \\ y = Cx + D_f f \end{cases} \quad (65)$$

where $g(t, x, u) \in \mathcal{R}^n$ is a nonlinear function satisfies the following conditions:

$$(i) |g(t, x, u)| < c[1 + |x|] \quad (66)$$

$$(ii) |g(t_1, x_1, u_1) - g(t_2, x_2, u_2)|^2 \leq \rho_1 |x_1 - x_2|^2 + \rho_2 \langle x_1 - x_2, g(t_1, x_1, u_1) - g(t_2, x_2, u_2) \rangle \quad (67)$$

where $\rho_1, \rho_2 \in \mathcal{R}, c > 0$.

Remark 4

The above assumptions for $g(t, x, u)$ imply that $\forall x_0 \in \mathcal{R}^n$, system (65) has a path-wise strong solution, and $g(t, x, u)$ is quadratic inner-bounded [32]. The constants ρ_1, ρ_2 can be positive, negative or zero. When $\rho_1 > 0$ and $\rho_2 = 0$, condition (67) is equivalent to the Lipschitz condition, which means Lipschitz nonlinear systems is a specific scenario of quadratic inner-bounded nonlinear systems.

For plant (65), we can construct the following augmented system by representing the mean of fault as a part of state:

$$\begin{cases} d\bar{x} = [\bar{A}\bar{x} + \bar{B}u + \bar{B}_d d + \bar{g}(t, x, u)]dt + \bar{W}x dw \\ y = \bar{C}\bar{x} \end{cases} \quad (68)$$

where $\bar{g}(t, x, u) = [g(t, x, u)^T \quad 0_{1 \times l_f} \quad 0_{1 \times l_f}]^T \in \mathcal{R}^{\bar{n}}$, and other symbols are the same as the above sections.

An unknown input observer in the following form can be designed for (68):

$$\begin{cases} d\bar{z} = [R\bar{z} + S\bar{B}u + (L_1 + L_2)y + S\bar{g}(t, \hat{x}, u)]dt \\ \hat{x} = \bar{z} + Hy \end{cases} \quad (69)$$

Let the estimation error be $\bar{e} = \bar{x} - \hat{x}$, and $\tilde{g} = (t, x, u) - \bar{g}(t, \hat{x}, u)$, we can derive the following error dynamic if conditions (11)-(14) hold:

$$d\bar{e} = (R\bar{e} + S\bar{B}_{d2}d_2 + S\tilde{g})dt + S\bar{W}x dw \quad (70)$$

Substituting (17) into system (65) and using the compensated measurement output y_c to replace the actual measurement y , the following closed-loop system can be established

$$\begin{cases} dx = [(A + BK)x - B_e\bar{e} + B_d d + g(t, x, u)]dt + Wx dw \\ d\bar{e} = (R\bar{e} + S\bar{B}_{d2}d_2 + S\tilde{g})dt + S\bar{W}x dw \\ y_c = Cx + D_e\bar{e} \end{cases} \quad (71)$$

Since Lipschitz condition (51) fail to capture the nonlinear features of $g(t, x, u)$, LMIs (58) and (59) are invalid for the robust fault tolerant control design of system (65). Hence, the design of observer-based fault tolerant control becomes more challenging and requires alternative techniques. For closed-loop system (71), we firstly design K as in the aforementioned ways in section III and IV. Then, we can employ the following theorem to achieve the stochastically input-to-state stability and the robustness requirement.

Theorem 3

For system (65), there exist an unknown input observer in the form of (69), and the tolerant control laws in the forms of (17) and (24), such that the closed-loop system (71) is stochastically

input-to-state stable and satisfies the robust performance index (26), if there exist positive definite matrices P, \bar{P}, Q , and \bar{Q} , matrix Y , positive scalars τ_1 and τ_2 , such that

$$\begin{bmatrix} \Theta_{11} & -PB_e + C^T D_e & P + \tau_1 \rho_2 I_n & 0 & PB_{d1} & PB_{d2} \\ * & \Theta_{22} & 0 & \bar{P}S + \tau_2 \rho_2 J_0^T J_0 & 0 & \bar{P}S\bar{B}_{d2} \\ * & * & -2\tau_1 I_n & 0 & 0 & 0 \\ * & * & * & -2\tau_2 I_{\bar{n}} & 0 & 0 \\ * & * & * & * & -\gamma_1^2 I_{l_{d1}} & 0 \\ * & * & * & * & * & -\gamma_2^2 I_{l_{d2}} \end{bmatrix} < 0 \quad (72)$$

and

$$\bar{P}S\bar{A} + \bar{A}^T S^T \bar{P} - Y\bar{C} - \bar{C}^T Y^T + 2a\bar{P} < 0 \quad (73)$$

where

$$\Theta_{11} = P(A + BK) + (A + BK)^T P + W^T P W + \bar{W}^T S^T \bar{P} S \bar{W} + 2\tau_1 \rho_1 I_n + Q + C^T C,$$

$$\Theta_{22} = \bar{P}S\bar{A} + \bar{A}^T S^T \bar{P} - Y\bar{C} - \bar{C}^T Y^T + 2\tau_2 \rho_2 J_0^T J_0 + \bar{Q} + D_e^T D_e,$$

$$Y = \bar{P}L_1;$$

$$a = \beta \theta_l, \theta_l = -\min\{\text{Re}[\lambda_i(A + BK)]\} > 0, i = \{1, 2, \dots, n\} \text{ and } \beta > 1.$$

One can thus calculate $L_1 = \bar{P}^{-1}Y$.

Proof

Choosing the Lyapunov function in the form of (30), and with the same proof manner of *Theorem 1*, we can know it satisfies condition (5) in *Lemma 1*. Taking the infinitesimal generator along the state trajectories of (71), by using Itô formula, it follows that:

$$\begin{aligned} \mathcal{L}V &= x^T [P(A + BK) + (A + BK)^T P + W^T P W + \bar{W}^T S^T \bar{P} S \bar{W}] x - 2x^T P B_e \bar{e} \\ &\quad + 2x^T P B_d d + 2x^T P g(t, x, u) + \bar{e}^T (\bar{P}R + R^T \bar{P}) \bar{e} + 2\bar{e}^T \bar{P} S \bar{B}_{d2} d_2 + 2\bar{e}^T \bar{P} S \tilde{g} \end{aligned} \quad (74)$$

Condition (67) implies that

$$\rho_1 x^T x + \rho_2 x^T g(t, x, u) - g^T(t, x, u) g(t, x, u) \geq 0 \quad (75)$$

We also have

$$\begin{aligned} \tilde{g}^T \tilde{g} &= |\bar{g}(t, x, u) - \bar{g}(t, \hat{x}, u)|^2 \\ &= \left\| \begin{bmatrix} g(t, x, u) - g(t, \hat{x}, u) \\ 0_{l_f \times 1} \\ 0_{l_f \times 1} \end{bmatrix} \right\|^2 \end{aligned}$$

$$\begin{aligned}
&= |g(t, x, u) - g(t, \hat{x}, u)|^2 \\
&\leq \rho_1 |x - \hat{x}|^2 + \rho_2 \langle x - \hat{x}, g(t, x, u) - g(t, \hat{x}, u) \rangle
\end{aligned} \tag{76}$$

Since

$$|x - \hat{x}|^2 = |J_0 \bar{e}|^2 = \bar{e}^T J_0^T J_0 \bar{e} \tag{77}$$

and

$$\langle x - \hat{x}, g(t, x, u) - g(t, \hat{x}, u) \rangle = \bar{e}^T J_0^T J_0 \tilde{g} \tag{78}$$

we can obtain

$$\tilde{g}^T \tilde{g} \leq \rho_1 \bar{e}^T J_0^T J_0 \bar{e} + \rho_2 \bar{e}^T J_0^T J_0 \tilde{g} \tag{79}$$

Hence, based on (75) and (79), for any positive scalars τ_1, τ_2 , we have

$$2\tau_1 (\rho_1 x^T x + \rho_2 x^T g(t, x, u) - g^T(t, x, u) g(t, x, u)) \geq 0 \tag{80}$$

and

$$2\tau_2 (\rho_1 \bar{e}^T J_0^T J_0 \bar{e} + \rho_2 \bar{e}^T J_0^T J_0 \tilde{g} - \tilde{g}^T \tilde{g}) \geq 0 \tag{81}$$

Adding (80) and (81) to the right side of (74), we can derive:

$$\begin{aligned}
\mathcal{L}V &\leq x^T [P(A + BK) + (A + BK)^T P + W^T P W + \bar{W}^T S^T \bar{P} S \bar{W} + 2\tau_1 \rho_1 I_n + Q] x \\
&\quad - 2x^T P B_e \bar{e} + 2x^T P B_d d + 2x^T (P + \tau_1 \rho_2 I_n) g(t, x, u) \\
&\quad + \bar{e}^T (\bar{P} R + R^T \bar{P} + 2\tau_2 \rho_1 J_0^T J_0 + \bar{Q}) \bar{e} + 2\bar{e}^T \bar{P} S \bar{B}_{d2} d_2 \\
&\quad + 2\bar{e}^T (\bar{P} S + \tau_2 \rho_2 J_0^T J_0) \tilde{g} - 2\tau_1 g^T g - 2\tau_2 \tilde{g}^T \tilde{g} - x^T Q x - \bar{e}^T \bar{Q} \bar{e} \\
&= [x^T \quad \bar{e}^T \quad g^T \quad \tilde{g}^T \quad d_1^T \quad d_2^T] \Phi \begin{bmatrix} x \\ \bar{e} \\ g \\ \tilde{g} \\ d_1 \\ d_2 \end{bmatrix} - x^T Q x - \bar{e}^T \bar{Q} \bar{e} + \gamma_1^2 d_1^T d_1 + \gamma_2^2 d_2^T d_2
\end{aligned} \tag{82}$$

where

$$\Phi = \begin{bmatrix} \Phi_{11} & -PB_e & P + \tau_1\rho_2 I_n & 0 & PB_{d1} & PB_{d2} \\ * & \Phi_{22} & 0 & \bar{P}S + \tau_2\rho_2 J_0^T J_0 & 0 & \bar{P}S\bar{B}_{d2} \\ * & * & -2\tau_1 I_n & 0 & 0 & 0 \\ * & * & * & -2\tau_2 I_{\bar{n}} & 0 & 0 \\ * & * & * & * & -\gamma_1^2 I_{l_{d1}} & 0 \\ * & * & * & * & * & -\gamma_2^2 I_{l_{d2}} \end{bmatrix}$$

$$\Phi_{11} = P(A + BK) + (A + BK)^T P + W^T P W + \bar{W}^T S^T \bar{P} S \bar{W} + 2\tau_1 \rho_1 I_n + Q$$

$$\Phi_{22} = \bar{P}S\bar{A} + \bar{A}^T S^T \bar{P} - \bar{P}L_1 \bar{C} - \bar{C}^T L_1^T \bar{P}^T + 2\tau_2 \rho_1 J_0^T J_0 + \bar{Q}.$$

From the LMI (72), one has

$$\Phi < 0 \quad (83)$$

which indicates we can find a positive scalar $\bar{\lambda}$ such that

$$\begin{aligned} \mathcal{L}V &\leq -x^T Q x - \bar{e}^T \bar{Q} \bar{e} + \gamma_1^2 d_1^T d_1 + \gamma_2^2 d_2^T d_2 \\ &\leq -\bar{\lambda} |\tilde{x}|^2 + \gamma_1^2 |d_1|^2 + \gamma_2^2 |d_2|^2 \end{aligned} \quad (84)$$

As a result, we can conclude the closed-loop system (71) is stochastically input-to-state sable with $\psi_3(\tilde{x}) = \bar{\lambda} |\tilde{x}|^2$ and $\psi_4(|d|) = \gamma_1^2 |d_1|^2 + \gamma_2^2 |d_2|^2$.

Now it is ready to discuss the robustness of the observer-based fault tolerant control:

$$\begin{aligned} \Gamma &= \mathbb{E}\{\int_0^{Tf} [y_c^T(\tau) y_c(\tau) - \gamma_1^2 d_1^T(\tau) d_1(\tau) - \gamma_2^2 d_2^T(\tau) d_2(\tau)] d\tau\} \\ &\leq \mathbb{E}\{\int_0^{Tf} \{[x^T \quad \bar{e}^T \quad g^T \quad \tilde{g}^T \quad d_1^T \quad d_2^T] \Theta \begin{bmatrix} x \\ \bar{e} \\ g \\ \tilde{g} \\ d_1 \\ d_2 \end{bmatrix} - x^T(\tau) Q x(\tau) - \bar{e}^T(\tau) \bar{Q} \bar{e}(\tau)\} d\tau\} \\ &\quad - \mathbb{E}(\int_0^{Tf} \mathcal{L}V d\tau) \end{aligned} \quad (85)$$

where

$$\Theta = \begin{bmatrix} \Theta_{11} & -PB_e + C^T D_e & P + \tau_1\rho_2 I_n & 0 & PB_{d1} & PB_{d2} \\ * & \Theta_{22} & 0 & \bar{P}S + \tau_2\rho_2 J_0^T J_0 & 0 & \bar{P}S\bar{B}_{d2} \\ * & * & -2\tau_1 I_n & 0 & 0 & 0 \\ * & * & * & -2\tau_2 I_{\bar{n}} & 0 & 0 \\ * & * & * & * & -\gamma_1^2 I_{l_{d1}} & 0 \\ * & * & * & * & * & -\gamma_2^2 I_{l_{d2}} \end{bmatrix},$$

$$\Theta_{11} = P(A + BK) + (A + BK)^T P + W^T P W + \bar{W}^T S^T \bar{P} S \bar{W} + 2\tau_1 \rho_1 I_n + Q + C^T C,$$

$$\Theta_{22} = \bar{P}S\bar{A} + \bar{A}^T S^T \bar{P} - Y\bar{C} - \bar{C}^T Y^T + 2\tau_2 \rho_1 J_0^T J_0 + \bar{Q} + D_e^T D_e, Y = \bar{P}L_1.$$

We know $\Theta < 0$ from the LMI (72) and $\mathbb{E}(\int_0^{Tf} \mathcal{L}V d\tau) > 0$, thus we have $\Gamma < 0$, which indicates the performance (26) can be satisfied. Similarly to *Theorems 1* and 2, the LMI (73) implies the response of the estimation error dynamics is faster than that of the system dynamics. This completes the proof.

Remark 5

Since Lipschitz nonlinear condition is a specific scenario of the quadratic inner-bounded one, LMIs (72) and (73) are also suitable for application in stochastic Lipschitz nonlinear system (50), by letting $\rho_1 = \theta^2$ and $\rho_2 = 0$. In other words, LMIs (72) and (73) are alternative rules of (58) and (59) for robust tolerant control of plant (50). Moreover, linear system is a special case of (65), where $g(t, x, u) = 0$. Hence, $|g(t_1, x_1, u_1) - g(t_2, x_2, u_2)|^2 = 0$. We find it satisfies (67) by letting $\rho_1 = 0$, and $\rho_2 = 0$. Then LMIs (72) and (73) are also suitable for application in stochastic linear system.

Now, the design procedure of the robust fault estimation and fault tolerant control strategies for stochastic quadratic inner-bounded nonlinear systems can be summarized as follows:

Procedure 3 (Fault tolerant control algorithm by integrating state/fault estimation and signal compensation for stochastic quadratic inner-bounded nonlinear systems)

- i) Construct the augmented system in the form of (68) for system (65).
- ii) Select observer gains H and S following step ii) in *Procedure 1*.
- iii) Design control gain K in the same way with step iii) in *Procedure 1*.
- iv) Solve the LMIs (72) and (73) to obtain P, Q, \bar{P}, \bar{Q} and matrix Y . The observer gain is thus calculated as $L_1 = \bar{P}^{-1}Y$.
- v) Calculate the other observer gains R and L_2 following the formulas (11) and (14), respectively.
- vi) Implement the robust unknown input observer (69) to produce the augmented estimate $\hat{\bar{x}}$, leading to simultaneous estimates of the system state and mean of fault (\hat{x} and \hat{f}) in the forms of (18) and (19), respectively.
- vii) Implement the tolerant control law $u = \bar{K}\hat{\bar{x}}$ and $y_c = y - D_f\hat{f}$, where $\bar{K} = [K \quad 0 \quad K_f]$ and $K_f = -B^+B_f$.

Remark 6

We have proposed integrated and robust fault estimation and fault-tolerant control algorithms with a strict input-to-state stability analysis, for linear systems, Lipschitz nonlinear

systems and quadratic inner-bounded nonlinear systems, respectively. The partially input uncertainties and Brownian parameter perturbations are considered, which can describe practical systems more practically and precisely so that the proposed method has a wide applicability. Recently, Takagi-Sugeno fuzzy systems and type-2 fuzzy systems were used to describe high-nonlinear systems [33-37]. It is interesting and encouraging to extend the current work to stochastic nonlinear systems which can be modelled by stochastic fuzzy model, which may even widen the applicability of the proposed diagnosis and tolerant control methods. This is regarded as a future work; whose research is under way.

6. SIMULATION STUDIES

In this session, three examples will be given to illustrate the effectiveness of the proposed fault tolerant control approaches.

Example 1 (Wind turbine drive train system)

A benchmark model for wind turbines was designed in [38], based on a generic three blade horizontal wind turbine driven by variable wind speeds, with a full converter coupling and a rated power of 4.8 MW. The drive train is responsible for increasing the rotational speed from the rotor to generator. The model includes low and high speed shafts linked together by a gearbox modeled as a gear ratio. When the system is affected by stochastic parameter perturbations, the state space model is given by:

$$\begin{cases} d\omega_r = \left(-\frac{B_{dt}+B_r}{J_r} \omega_r + \frac{B_{dt}}{N_g J_r} \omega_g - \frac{K_{dt}}{J_r} \theta_\Delta + \frac{1}{J_r} \tau_r \right) dt + 0.02 \omega_r dw \\ d\omega_g = \left(\frac{\eta_{dt} B_{dt}}{N_g J_g} \omega_r + \frac{-\frac{\eta_{dt} B_{dt}}{N_g^2} - B_g}{J_g} \omega_g + \frac{\eta_{dt} K_{dt}}{N_g J_g} \theta_\Delta - \frac{1}{J_g} \tau_g \right) dt + 0.05 \omega_g dw \\ d\theta_\Delta = \left(\omega_r - \frac{1}{N_g} \omega_g \right) dt + 0.01 \theta_\Delta dw \\ y_1 = \omega_r \\ y_2 = \omega_g \end{cases} \quad (86)$$

where the meanings of parameters are shown in Table 1.

Table 1. Parameter symbols of benchmark wind turbines

ω_r	Rotor angular speed	K_{dt}	Torsion stiffness
θ_Δ	Torsion angle	N_g	Gear ratio

ω_g	Generator rotating speed	J_r	Rotor moment of inertia
B_g	Generator external damping	J_g	Generator moment of inertia
τ_r	Aerodynamic torque	η_{dt}	Efficiency of drive train
B_r	Rotor external damping	τ_g	Generator torque
$\tau_{g,r}$	Generator torque reference	B_{dt}	Torsion damping coefficient

When the plant (86) is subject to faults and unknown input disturbances, it can be represented by system (7) with $x = [\omega_r \ \omega_g \ \theta_\Delta]^T$, and the initial value $x_0 = [1 \ 100 \ 0]^T$ corrupted by random noises; $u = [\tau_r \ \tau_g]^T$, where τ_r and τ_g take references from pitch system and generator system of the benchmark wind turbine, respectively. We consider $d = [d_1 \ d_2 \ d_3]^T$, and d_1, d_2 and d_3 are random signals with range from -10^{-2} to 10^{-2} ; actuator fault f_a is 50% loss of actuation effectiveness for τ_g from 2000 seconds to 3500 seconds and the coefficients are as follows:

$$A = \begin{bmatrix} -\frac{B_{dt}+B_r}{J_r} & \frac{B_{dt}}{N_g J_r} & -\frac{K_{dt}}{J_r} \\ \frac{\eta_{dt} B_{dt}}{N_g J_g} & \frac{-\eta_{dt} B_{dt} - B_g}{N_g^2 J_g} & \frac{\eta_{dt} K_{dt}}{N_g J_g} \\ 1 & -\frac{1}{N_g} & 0 \end{bmatrix}, B = \begin{bmatrix} \frac{1}{J_r} & 0 \\ 0 & -\frac{1}{J_g} \\ 0 & 0 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, W = \begin{bmatrix} 0.02 & 0 & 0 \\ 0 & 0.05 & 0 \\ 0 & 0 & 0.01 \end{bmatrix},$$

$$B_f = \begin{bmatrix} 0 \\ -\frac{1}{J_g} \\ 0 \end{bmatrix}, B_d = \begin{bmatrix} -0.6 & 0.04 & -0.08 \\ 0.2 & -0.02 & 0.04 \\ -0.3 & -0.01 & 0.02 \end{bmatrix} \text{ and } D_f = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Since the drive train system is already a stable dynamic with desired response, we let control gain K to be zero. Selecting $\gamma_1 = 10$ and $\gamma_2 = 5$, and solving LMIs (28) and (29), the observer gain L_1 can be calculated as

$$L_1 = \begin{bmatrix} 539.86 & -2271.4 \\ -1618.7 & 6827.8 \\ 229.79 & -1047.9 \\ -14181 & 24558 \\ 11792 & 25226 \end{bmatrix}$$

Therefore R and L_2 can be obtained following the formulas (12) and (14), respectively.

Using the Euler–Maruyama method [39] to simulate the standard Brownian motion, one can obtain the simulated curves of the stochastic state responses (here we give 5 state trajectories). Figures 1-4 exhibit the estimation performance for full system states and the mean of actuator fault, respectively. Figures 5 and 6 compare the system outputs with and without tolerant control, and healthy outputs in fault-free cases.

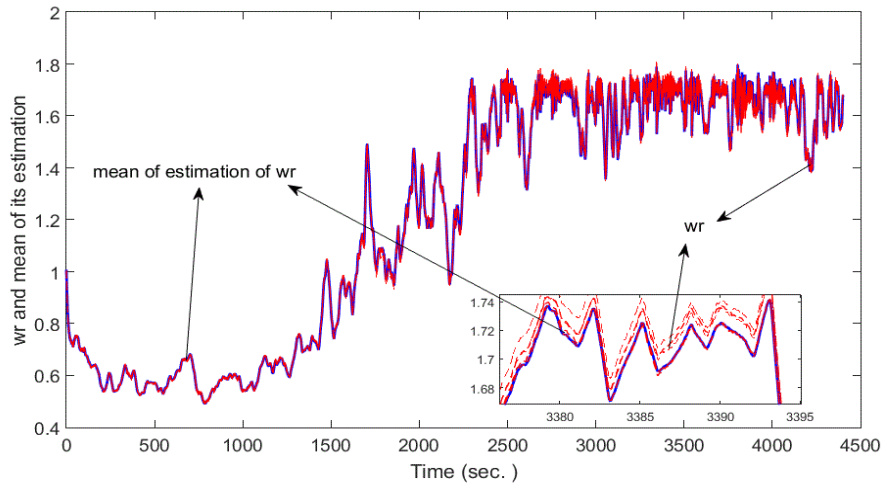


Figure 1. Rotor angular speed and the mean of its estimate: wind turbine.

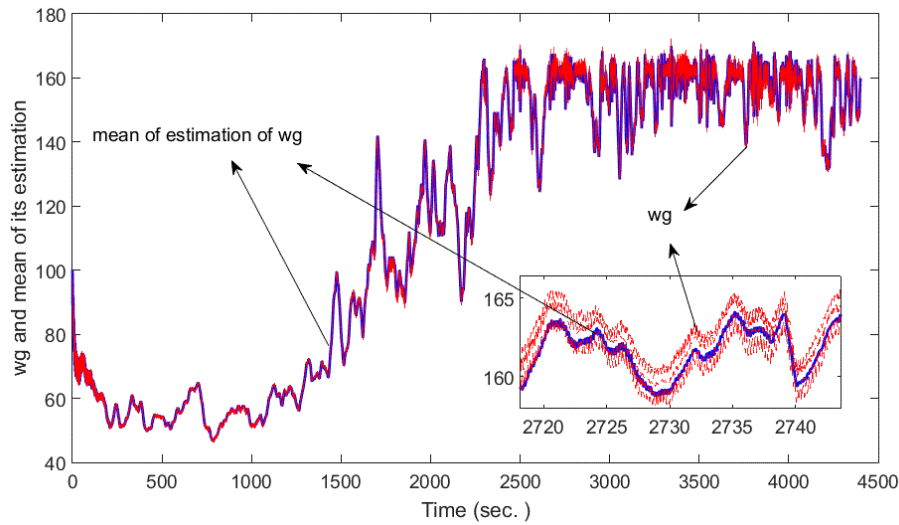


Figure 2. Generator rotating speed and the mean of its estimate: wind turbine.

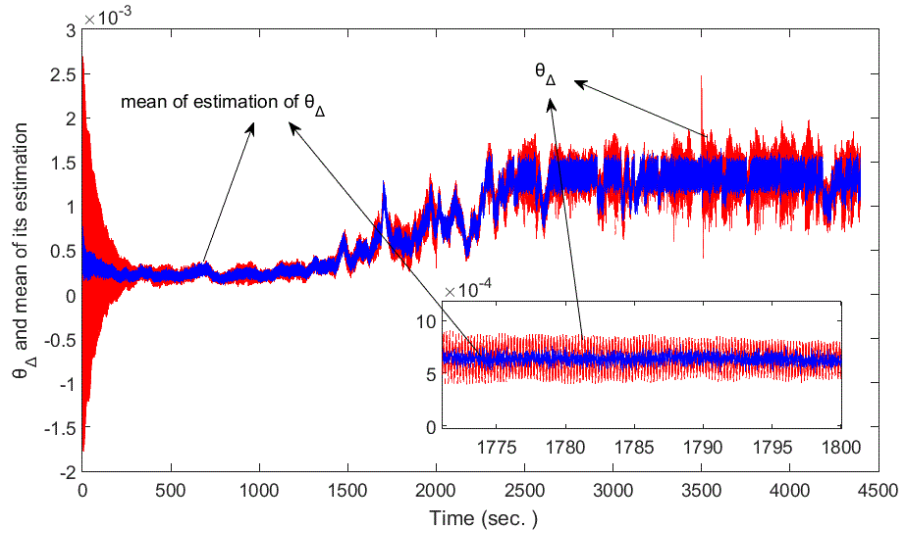


Figure 3. Torsion angle and the mean of its estimate: wind turbine.

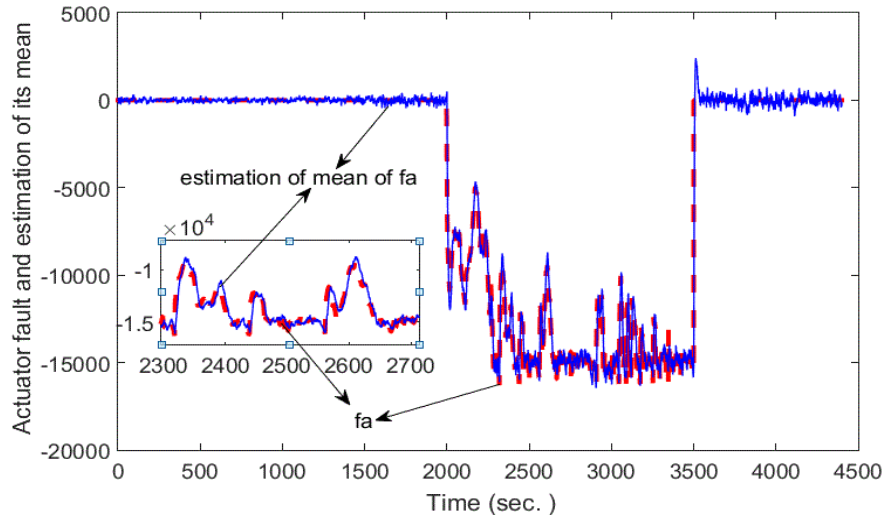


Figure 4. Generator torque fault and the estimate of its mean: wind turbine.

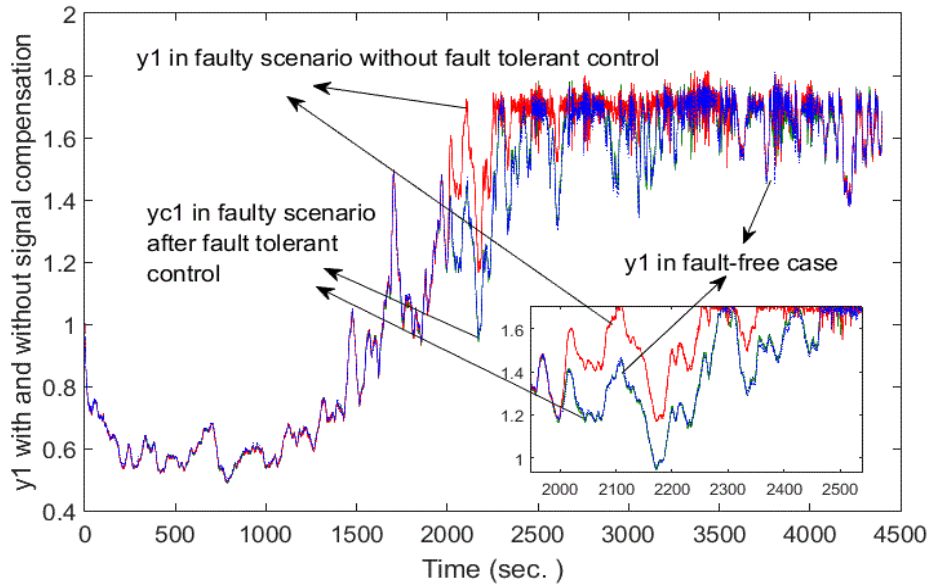


Figure 5. Comparisons of the first output: fault-free output y_1 , output y_1 subjected to faults without tolerant control, and output subjected to faults after tolerant control denoted by y_{c1} : wind turbine.

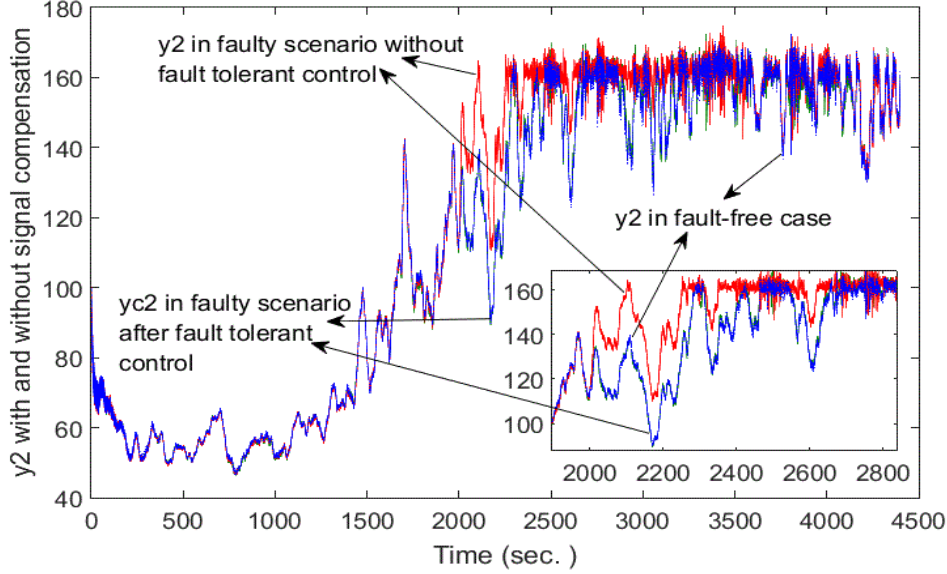


Figure 6. Comparisons of the second output: fault-free output y_2 , output y_2 subjected to faults without tolerant control, and output subjected to faults after tolerant control denoted by y_{c2} : wind turbine.

From the Figures 1-4, we can see that both system states and the mean of actuator fault are estimated satisfactorily, and the influences of the unknown inputs are decoupled/attenuated successfully. Moreover, we can find in Figures 5 and 6 that the actuator fault will make the deviation of the outputs. However, after tolerant control, the deviation is eliminated/offset successfully, as one can see the compensated outputs are consistent with the fault-free outputs. As a result, the proposed fault estimation-based fault tolerant control techniques are effective.

Example 2 (Three-tank system)

Considered the three-tank system modelled in [40] affected by the Brownian motion and nonlinear perturbation term:

$$\begin{cases} dh_1 = \left[-\frac{1}{s_a} az_1 S_n \sqrt{2g(h_1^* - h_3^*)} + \frac{1}{s_a} q_1 \right] dt + 0.03h_1 dw \\ dh_2 = \left[\frac{1}{s_a} az_3 S_n \sqrt{2g(h_3^* - h_2^*)} - \frac{1}{s_a} az_2 S_n \sqrt{2gh_2^*} + \frac{1}{s_a} q_2 \right] dt + 0.01h_2 dw \\ dh_3 = \left[\frac{1}{s_a} az_1 S_n \sqrt{2g(h_1^* - h_3^*)} - \frac{1}{s_a} az_3 S_n \sqrt{2g(h_3^* - h_2^*)} + \frac{1}{s_a} q_3 + 0.05 \sin(h_3) \right] dt \\ \quad + 0.05h_3 dw \end{cases} \quad (87)$$

where $\forall i = 1, 2, 3$, h_i represent the liquid level (in m) of the three tanks; q_i are supplying flow rates (in m^3/s) of the three pumps; az_i are outflow coefficients taking values of 0.48, 0.5 and 0.58, respectively; $S_a = 0.0154m^2$ and $S_n = 5 \times 10^{-5}m^2$ are the cross sections; $[h_1^* \ h_2^* \ h_3^*]^T = [0.4890 \ 0.2332 \ 0.3611]^T$ is an equilibrium point under a nominal control law which is not in our concern.

When the system is subjected to faults and unknown inputs, it can be represented by plant (50), where $x = [h_1 \ h_2 \ h_3]^T$ with initial value $x_0 = [0.4890 \ 0.2332 \ 0.3611]^T$ corrupted by random noises; $u = [q_1 \ q_2 \ q_3]^T$ with reference input of $[38 \times 10^{-6} \ 24 \times 10^{-6} \ 0]^T$; $d = [d_1 \ d_2 \ d_3]^T$, d_1 , d_2 and d_3 are random signals with range from -10^{-6} to 10^{-6} ; $f = [f_{a1} \ f_{a2}]^T$, and actuator faults f_{a1} and f_{a2} are 50% loss of actuation effectiveness for pump 1 (from 25sec. to 50 sec.) and pump 2 (from 60sec. to 100sec.), respectively; $\phi(x) = [0 \ 0 \ 0.05 \sin(x_3)]^T$, and the coefficients are as follows:

$$A = \begin{bmatrix} -0.0096 & 0 & 0.0096 \\ 0 & 0.0042 & 0.0117 \\ 0.0096 & 0.0117 & -0.0020 \end{bmatrix}, B = \begin{bmatrix} 64.935 & 0 & 0 \\ 0 & 64.935 & 0 \\ 0 & 0 & 64.935 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, W = \begin{bmatrix} 0.03 & 0 & 0 \\ 0 & 0.01 & 0 \\ 0 & 0 & 0.05 \end{bmatrix},$$

$$B_f = \begin{bmatrix} 64.935 & 0 \\ 0 & 64.935 \\ 0 & 0 \end{bmatrix}, B_d = \begin{bmatrix} 0.3 & -0.1 & 0.05 \\ -0.1 & -0.2 & 0.1 \\ -0.2 & -0.04 & 0.02 \end{bmatrix} \text{ and } D_f = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

It is obvious that

$$|\phi(x)| \leq 0.05|x - \hat{x}| \quad (88)$$

So $\phi(x)$ satisfies Lipschitz condition with $\theta = 0.05$. Select $\mathcal{D}(c, \mu, \delta) = \mathcal{D}(0.25, 2, 60)$ as the region of poles for the plant after implementing control input. Solving LMIs (47) to (49), we can obtain observer gain

$$K = \begin{bmatrix} -0.0156 & 0 & -0.0001 \\ 0 & -0.0158 & -0.0002 \\ -0.0001 & -0.0002 & -0.0157 \end{bmatrix}$$

and $\min\{\text{Re}[\lambda_i(A + BK)]\} = -1.0235$. Let $\beta = 2.5$, then $a = 2.5587$. Selecting $\gamma_1 = 1$ and $\gamma_2 = 0.5$, and substituting θ , K and a to the LMIs (58) and (59), the observer gain L_1 can be calculated as

$$L_1 = \begin{bmatrix} 38.0427 & 97.8116 & -3.9286 \\ 106.443 & 347.199 & 8.4629 \\ 1.4897 & 25.4615 & 13.7447 \\ 67.869 & 31.2794 & -86.7459 \\ 43.5794 & 193.563 & 28.1524 \\ 19.2956 & 11.5666 & -23.3783 \\ 16.9729 & 70.1258 & 8.4585 \end{bmatrix}$$

Then R and L_2 can be obtained following the formulas (12) and (14), respectively.

Using the Euler–Maruyama method to simulate the standard Brownian motion, one can obtain the simulated curves of the stochastic state responses (here we give 15 state trajectories).

We employ the controller $u = \bar{K}_0 \hat{x} + K_p(Cx^* - y_c)$, where K_p is selected as

$$\begin{bmatrix} 0.0158 & 0 & 0 \\ 0 & 0.0158 & 0 \\ 0 & 0 & 0.0158 \end{bmatrix} \text{ to drive the trajectory of the state to the equilibrium point } x^*,$$

$\bar{K}_0 = [K_0 \ 0 \ K_f]$, and $K_0 = K + K_p C$. The curves displayed in Figures 7 and 8 exhibit the estimation performance for full system states and the means of actuator faults after implementing the designed estimator-based fault tolerant control method, respectively. Figures 9 and 10 show the system outputs with signal compensation and without signal compensation.

From the Figures 7 and 8, we can see that both system states and the means of two actuator faults can be estimated robustly, and the influences of unknown inputs have been attenuated successfully. By comparison of Figures 9 and 10, we can find the actuator faults will make deviation of the liquid levels, however, based on accurate estimates of faults, the deviation can be compensated by the proposed fault tolerant control strategy. The differences of tank 1 and tank 2 are more distinguished than tank 3 because the loss of actuation effectiveness occurs in tank 1 and tank 2, hence influences more on the liquid levels of tank 1 and 2 than that of tank 3.

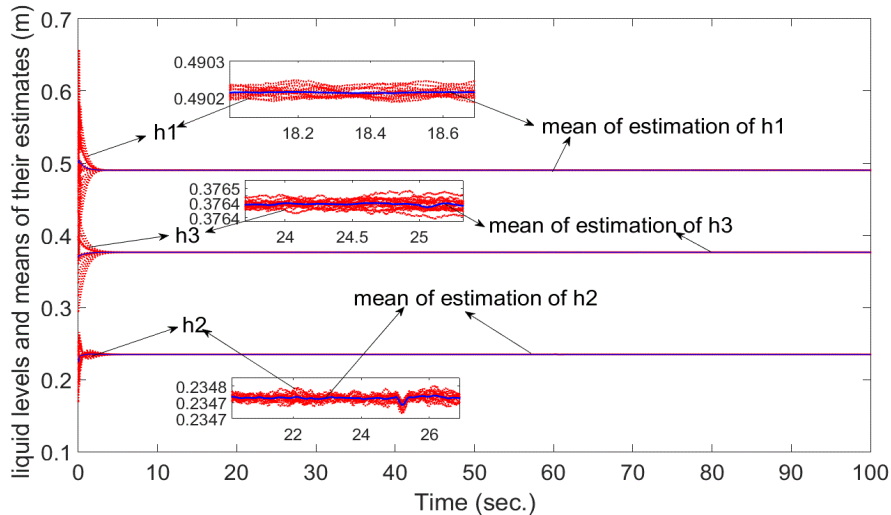


Figure 7. System states and means of their estimates: three-tank system.

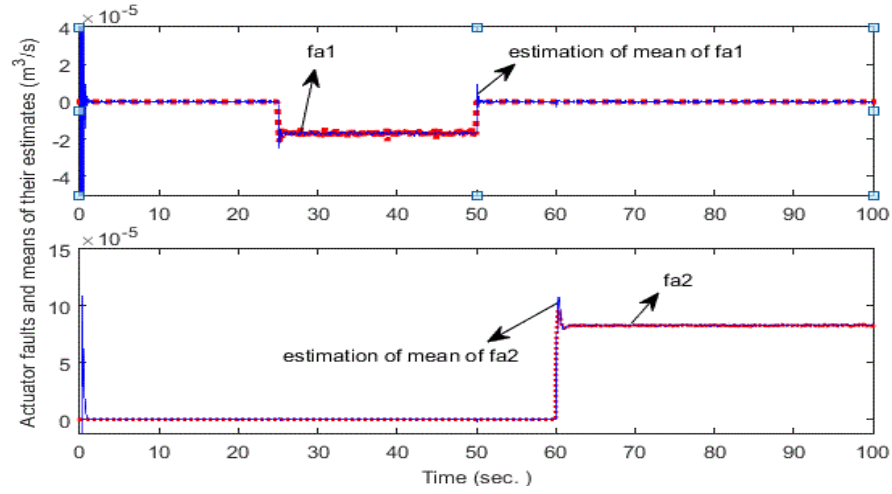


Figure 8. Actuator faults and estimates of their means: three-tank system.

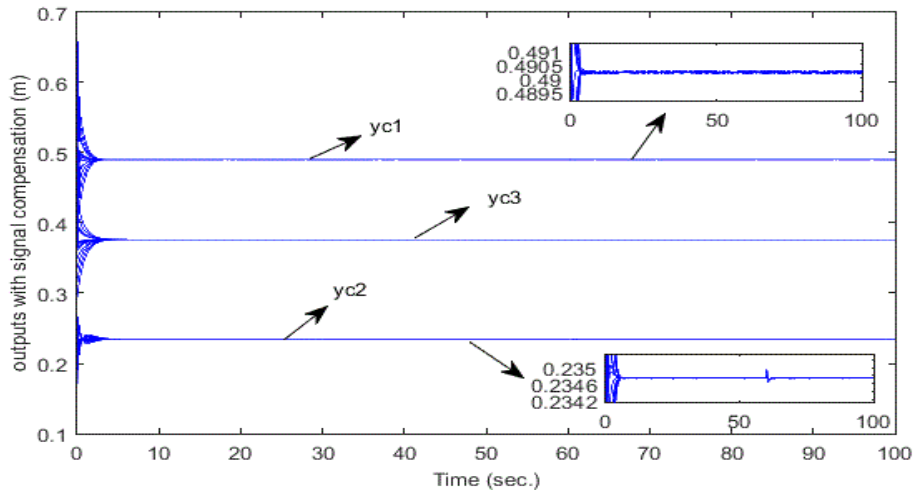


Figure 9. System outputs with signal compensation: three-tank system.

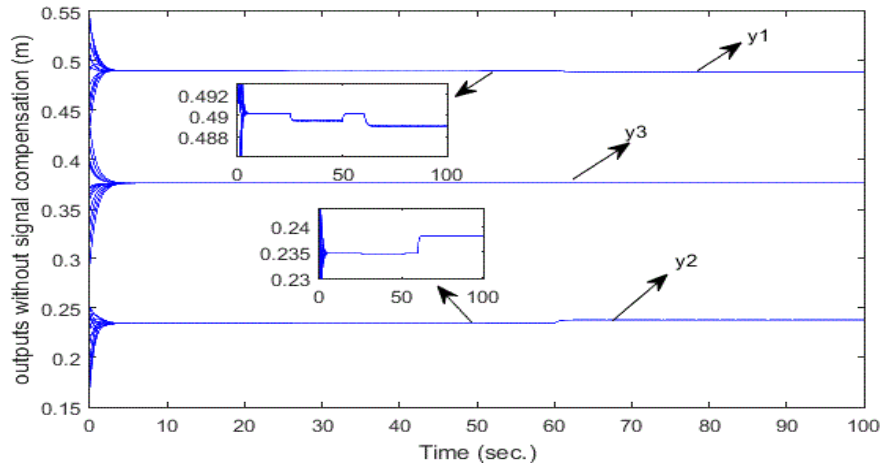


Figure 10. System outputs without signal compensation: three-tank system.

Example 3 (A quadratic inner-bounded nonlinear system)

In this example, we apply the robust observer-based controller to quadratic inner-bounded nonlinear system. The considered plant is in the form of (65) with the following parameters:

$$A = \begin{bmatrix} 1 & 3 & 1 \\ 2 & 0 & 3 \\ 0 & 2 & 5 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, W = \begin{bmatrix} 0.03 & 0 & -0.02 \\ 0 & 0.01 & 0.04 \\ 0.05 & 0 & 0.01 \end{bmatrix},$$

$$B_d = \begin{bmatrix} 0.3 & 0.1 & -0.02 \\ -0.1 & -0.2 & 0.04 \\ -0.15 & -0.4 & 0.08 \end{bmatrix}, g(x) = \begin{bmatrix} -0.1x_1(x_1^2 + x_2^2 + x_3^2) \\ -0.1x_2(x_1^2 + x_2^2 + x_3^2) \\ -0.1x_3(x_1^2 + x_2^2 + x_3^2) \end{bmatrix}, B_{f_a} = B, D_{f_s} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

In this case, $B_f = [B_{f_a} \ 0_{3 \times 1}]$, $D_f = [0_{3 \times 1} \ D_{f_s}]$. The reference input is given as 0.5. The actuator fault and the sensor fault are 50% loss of the actuation effectiveness from 25 sec. to 50 sec., and 30% loss of the sensor efficiency from 70 sec. to 100 sec., respectively, and the unknown input disturbances are random signals ranging from -0.01 to 0.01 . The initial state value is given as $x_0 = [-0.1 \ -0.05 \ -0.2]^T$ corrupted by random noises. Considering the set $\tilde{D} = \{x \in R^3: |x| \leq \vartheta\}$, we have $|g(x)| < 0.1\vartheta^2[1 + |x|]$. After some algebraic manipulations, we can obtain

$$|g(x_1) - g(x_2)|^2 = 0.01[(|x_1|^2 - |x_2|^2)^2(|x_1|^2 + |x_2|^2) + |x_1 - x_2|^2|x_1|^2|x_2|^2] \quad (89)$$

$$\begin{aligned} \rho_1 |x_1 - x_2|^2 + \rho_2 \langle x_1 - x_2, g(x_1) - g(x_2) \rangle &= |x_1 - x_2|^2 \left[\rho_1 - \frac{0.1\rho_2}{2}(|x_1|^2 + |x_2|^2) \right] \\ &\quad - \frac{0.1\rho_2}{2}(|x_1|^2 - |x_2|^2)^2 \end{aligned} \quad (90)$$

In order to make (67) hold, we have to find ρ_1 and ρ_2 such that

$$|x_1|^2 + |x_2|^2 \leq -5\rho_2 \quad (91)$$

$$|x_1|^2 \cdot |x_2|^2 \leq 100\rho_1 + 25\rho_2^2 \quad (92)$$

hold in set \tilde{D} . It suffices to have $\rho_2 \leq -0.4\vartheta^2$ and $\rho_1 \geq 0.01\vartheta^4 - 0.25\rho_2^2$. For given set \tilde{D} with $\vartheta = 1.04$, which is large enough in terms of the considered system, we can find $\rho_1 = -0.0373$ and $\rho_2 = -0.44$ to make $g(x)$ satisfy the quadratic inner-bounded condition. Select $\mathcal{D}(c, \mu, \delta) = \mathcal{D}(0.2, 2.8, 55)$ as the region of poles for the original plant after implementing control input. Solving LMIs (47) to (49), we can obtain observer gain $K = [-4.7532 \ -6.9039 \ -10.8230]$, and $\min\{\text{Re}[\lambda_i(A + BK)]\} = -2.1324$. Let $\beta = 2$, then

$a = 4.2648$. Selecting $\gamma_1 = 10$ and $\gamma_2 = 20$, and substituting ρ_1, ρ_2, K and a to LMIs (72) and (73), L_1 can be calculated as

$$L_1 = 10^3 \times \begin{bmatrix} 0.1800 & 2.0167 & 0.4592 \\ 0.0182 & 0.2160 & 0.0548 \\ -0.0061 & 0.0346 & 0.0632 \\ -0.1015 & 3.3422 & 2.7269 \\ -0.4043 & -4.8093 & -1.1991 \\ -0.2256 & -1.2118 & 0.3026 \\ -0.1653 & -1.9653 & -0.4700 \end{bmatrix}$$

Then R and L_2 can be obtained following the formulas (12) and (14), respectively.

By choosing the above parameters, and using the Euler–Maruyama method to simulate the standard Brownian motions with 10 state trajectories, we can obtain Figures 11 and 12 to exhibit the estimation performance for full system states, the means of actuator fault and sensor fault, respectively. Figures 13 and 14 show the system output without and with signal compensation, respectively.

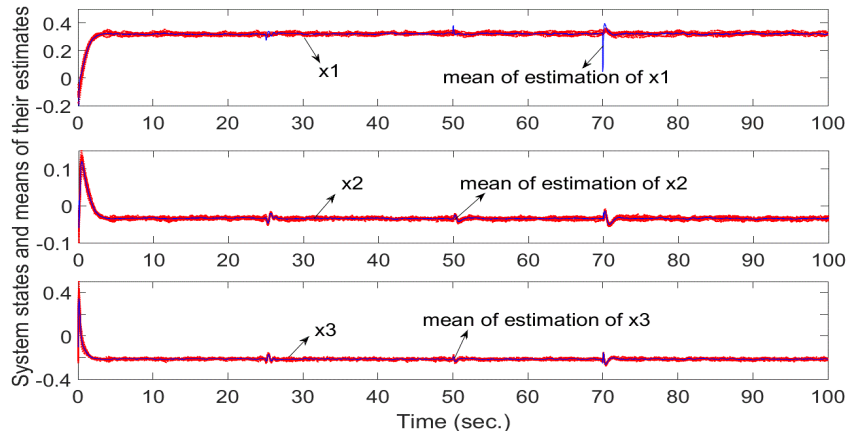


Figure 11. System states and the means of their estimates:
a quadratic inner-bounded nonlinear system.

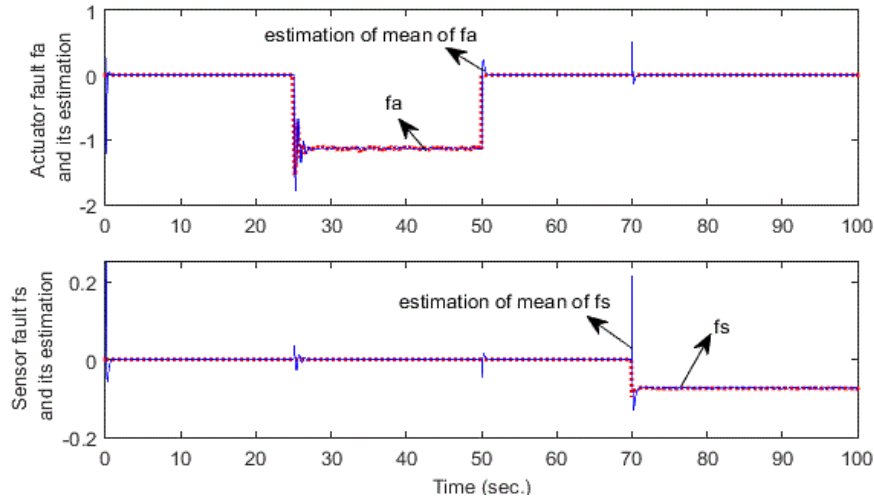


Figure 12. Faults and the estimates of their means:
a quadratic inner-bounded nonlinear system.

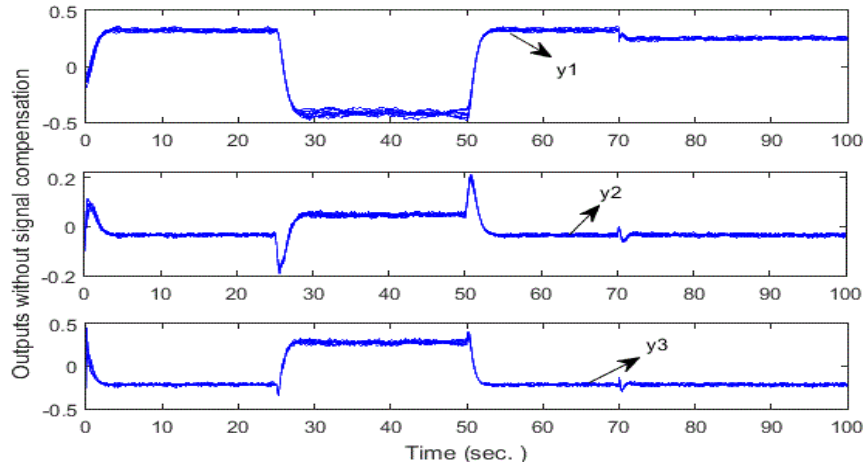


Figure 13. System outputs without tolerant control:
a quadratic inner-bounded nonlinear system.

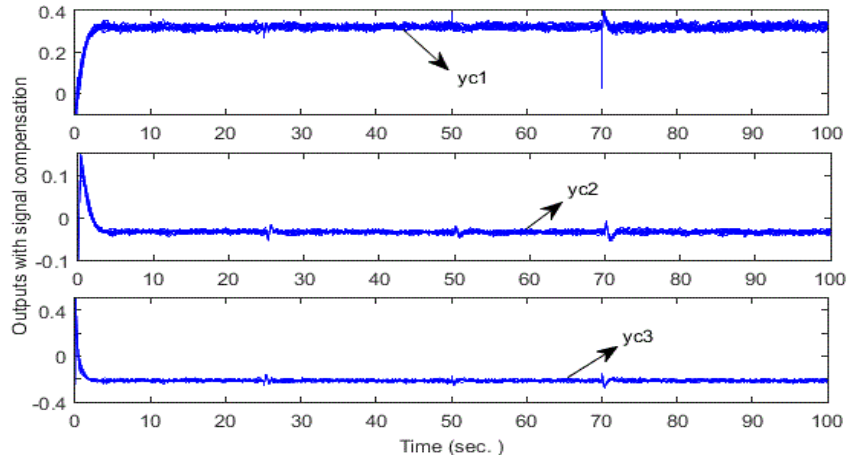


Figure 14. System outputs with tolerant control:
a quadratic inner-bounded nonlinear system.

From Figures 11 and 12, system states and the means of concerned faults are estimated satisfactorily. Figure 13 shows the system dynamics are corrupted by the faults, while Figure 14 indicates the system performances are recovered after the fault-tolerant control.

7. CONCLUSION

In this study, an integrated and robust fault estimation and tolerant control technique has been developed for stochastic systems subjected to Brownian parameter perturbations, partially decoupled unknown inputs, and unexpected faults. The proposed approach has integrated augmented system approach, UIO, LMI optimization, actuator and sensor signal compensation

techniques and input-to-state stability principle. The design of tolerant control for linear systems, Lipchitz nonlinear systems and quadratic inner-bounded nonlinear systems, corrupted by Brownian parameter perturbations and partially decoupled input disturbances, are investigated respectively. The effectiveness of the proposed fault tolerant control algorithms has been well demonstrated by the simulation studies on three examples. It is of interest to extend the proposed methods/algorithms to more complex systems such as stochastic fuzzy Brownian systems, and the research on this topic is under way.

REFERENCES

1. Chen J, Patton RJ. *Robust model-based fault diagnosis for dynamic systems*. Boston: Kluwer Academic Publishers, 1999.
2. Blanke M, Kinnaert M, Lunze J, Staroswiecki M. *Diagnosis and fault-tolerant control*. Berlin: Springer, 2006.
3. Gao Z, Cecati C, Ding SX. A Survey of fault diagnosis and fault-tolerant techniques part I: fault diagnosis with model-based and signal-based approaches. *IEEE Trans. Ind. Electron.* 2015; **62**(6):3757-3767.
4. Simani S, Farsoni S, Castaldi P. Fault diagnosis of a wind turbine benchmark via identified fuzzy models. *IEEE Trans. Ind. Electron.* 2015; **62**(6):3775-3782.
5. Cugueró-Escofet MÀ, Puig V, Quevedo J. Optimal pressure sensor placement and assessment for leak location using a relaxed isolation index: Application to the Barcelona water network. *Control Engineering Practice* 2017; **63**:1-12.
6. Bessaoudi T, Hmida F, Hsieh C. Robust state and fault estimation for linear descriptor stochastic systems with disturbances: a DC motor application. *IET Control Theory Appl.* 2017; **11**(5):601-610.
7. Mousavi SH, Marquez HJ. Integral-based event triggering controller design for stochastic LTI systems via convex optimisation. *International Journal of Control* 2016; **89**(7):1416-1427.
8. Li H, Shi P, Yao D. Adaptive sliding-mode control of Markov jump nonlinear systems with actuator faults. *IEEE Trans. Automat. Contr.* 2017; **62**(4):1933-1939.
9. Zhang QL, Li L, Yan XG, Spurgeon SK. Sliding mode control for singular stochastic Markovian jump systems with uncertainties. *Automatica* 2017; **79**:27-34.

10. Yao L, Qin J, Wang H, Jiang B. Design of new fault diagnosis and fault tolerant control scheme for non-Gaussian singular stochastic distribution systems. *Automatica* 2012; **48**(9):2305-2313.
11. Liu M, Cao X, Shi P. Fuzzy-model-based fault tolerant design for nonlinear stochastic systems against simultaneous sensor and actuator faults. *IEEE Tran. Fuzzy Syst.* 2013; **21**(5):789-799.
12. Liu M, Shi P. Sensor fault estimation and tolerant control for Itô stochastic systems with a descriptor sliding mode approach. *Automatica* 2013; **49**:1242-1250.
13. Mao X. *Stochastic Differential equations and applications*. London: Woodhead Publishing, 1997.
14. Gao Z, Shi X. Observer-based controller design for stochastic descriptor systems with Brownian motions. *Automatica* 2013; **49**(7):2229-2235.
15. Gao Z, Breikin T, Wang H. High-gain estimator and fault-tolerant design with application to a gas turbine dynamic system. *IEEE Trans. Contr. Tech.* 2007; **15**(4):740-753.
16. Zhang Q, Besancon G. An adaptive observer for sensor fault estimation in a class of uniformly observable non-linear systems. *Int. J. Mod. Indent. Contr.* 2008; **4**(1):37-43.
17. Jiang B, Staroswiecki M, Cocquempot V. Fault accommodation for a class of nonlinear systems, *IEEE Trans. Autom. Contr.* 2006; **51**(9): 1578-1583.
18. Han X, Emilia F, Spurgeon SK. Sampled-data sliding mode observer for robust fault reconstruction: A time-delay approach. *J. Franklin Inst.* 2014; **351**(4):2125-2142.
19. Alwi H, Edwards C. An adaptive sliding mode differentiator for actuator oscillatory failure case reconstruction. *Automatica* 2013; **49**:642-651.
20. Huang SJ, Yang GH. Input-output based fault estimation for linear systems with partially dynamic uncertainty and actuator faults. *International Journal of Robust and Nonlinear Control* 2016; **26**(16):3611-3630.
21. Menon PP, Edwards C. Robust fault estimation using relative information in linear multi-agent networks, *IEEE Transactions on Automatic Control* 2013; **59**(2):477-482.
22. Chen J, Zhang H. Robust fault detection of faulty actuators via unknown input observers. *Int. J. Syst. Sci.* 1991; **22**(10):1829-1839.
23. Gao Z, Liu X, Chen ZQ. Unknown input observer-based robust fault estimation for systems corrupted by partially decoupled disturbances. *IEEE Trans. Ind. Electron.* 2016; **63**(4):2537-2547.

24. Witczak M, Buciakowski M, Puig V, Rotondo D, Nejjari F. An LMI approach to robust fault estimation for a class of nonlinear systems. *International Journal of Robust and Nonlinear Control* 2016; **26**(7):1530-1548.
25. Liu X, Gao Z. Robust finite-time fault estimation for stochastic nonlinear systems with Brownian motions. *Journal of the Franklin Institute* 2017; **354**(6):2500-2523.
26. Tsinias J. Stochastic input-to-state stability and applications to global feedback stabilization. *Int. J. Contr.* 1998; **71**(5):907-930.
27. Hahn W. *Stability of motion*. Berlin: Springer-Verlag, 1967.
28. Liu S, Zhang J, Jiang Z. A notion of stochastic input-to-state stability and its application to stability of cascaded stochastic nonlinear systems. *Acta Math. Appl. Sin., Engl. Ser.* 2008; **24**(1):141-156.
29. Chilali M, Gahinet P. H_∞ design with pole placement constraints: an LMI approach. *IEEE Trans. Automat. Contr.* 1996; **41**(3):358-367.
30. Li L, Jia Y, Du J, Yuan S. Robust control for uncertain singular systems with time-varying delay. *Prog. Nat. Sci.* 2008; **18**(8):1015-1021.
31. Boyd S, Ghaoui LE, Feron E, Balakrishnan V. *Linear matrix inequalities in system and control theory*. SIAM: Philadelphia PA, 1994.
32. Beikzadeh H, Marquez HJ. Input-to-error stable observer for nonlinear sampled-data systems with application to one-sided Lipschitz systems. *Automatica* 2016; **67**:1-7.
33. Jiang B, Zhang K, Shi P. Integrated Fault Estimation and Accommodation Design for Discrete-Time Takagi–Sugeno Fuzzy Systems With Actuator Faults, *IEEE Trans. Fuzzy Syst.* 2011; **19**(2): 291-304.
34. Liu X, Gao Z, Chen ZQ. Takagi-Sugeno fuzzy model based fault estimation and signal compensation with application to wind Turbines, *IEEE Trans. Ind. Electron.* 2016; **64**(7): 5678-5689.
35. Pan Y, Yang G. Event-triggered fuzzy control for nonlinear networked control systems, *Fuzzy Sets and Systems* 2017; DOI: 10.1016/j.fss.2017.05.010.
36. Pan Y, Yang G. Switched filter design for interval type-2 fuzzy systems with sensor nonlinearities, *Neurocomputing* 2016; **194**:168-175.
37. Pan Y, Yang G. Event-Triggered Fault Detection Filter Design for Nonlinear Networked Systems, *IEEE Trans. Syst., Man, Cybern., Syst.* 2017; 329:91-107.
38. Odgaard PF, Stoustrup J, Nielsen R, Damgaard C. Observer based detection of sensor faults in wind turbines. *Proc. Eur. Wind Energy Conf.*, Yokohama, Japan, 2009; 4421-4430.

39. Higham D. An algorithmic introduction to numerical simulation of stochastic differential equations. *SIAM Review* 2001; **43**(3):525-546.
40. Zhou D, He X, Wang Z, Liu GP, Ji YD. Leakage fault diagnosis for an internet-based three-tank system: An experimental study. *IEEE Trans. Control Syst. Technol.* 2012; **20**(4):857-870.